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1. $A = \begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}$

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2/5 & 1/5 \\ -1/10 & 1/5 \end{pmatrix}$$

Given

$$\frac{1}{2}y + 2 = \frac{1}{2}x$$
$$y + 4 = x$$
$$x - y = 4$$
$$2x - 2y = 8 \quad \text{——— (1)}$$

Given

$$y = -\frac{1}{4} - \frac{1}{4}x$$
$$4y = -1 - x$$
$$x + 4y = -1 \quad \text{——— (2)}$$

Using (1), (2):

$$\begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ -1 \end{pmatrix}$$
$$= \frac{1}{10}\begin{pmatrix} 30 \\ -10 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$\therefore x = 3, \ y = -1$ #

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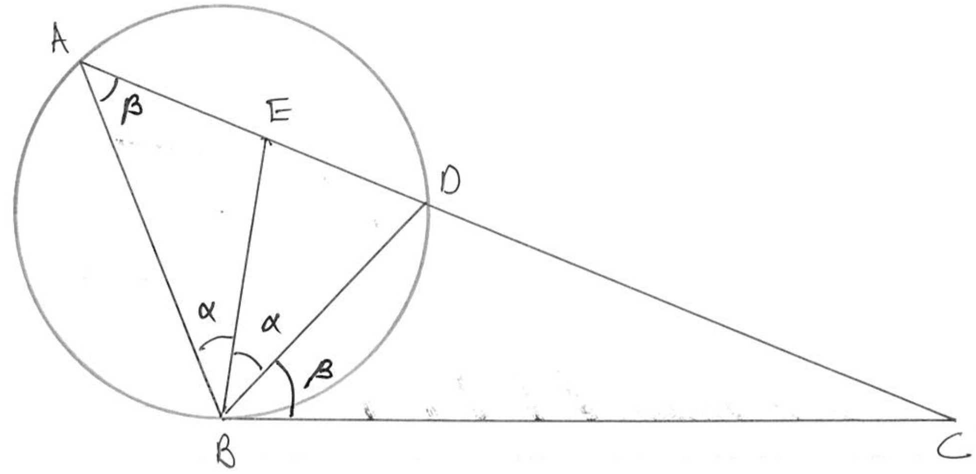
2.



Let $\angle DBE = \alpha$

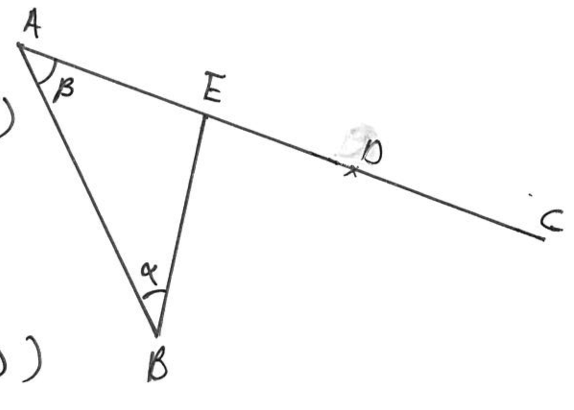
Given $\angle ABE = \angle DBE = \alpha$.

Let $\angle DBC = \beta$

Then $\angle CBE = \angle DBC + \angle DBE = \beta + \alpha$. —— ①.

$\angle BAD = \angle DBC = \beta$ ($\angle$ In alt segment).

$\angle BEC = \angle BAD + \angle ABE$
{ ext $\angle$ = sum of 2 Int opp $\angle$s)
$= \beta + \alpha$. —— ②.



Hence in $\triangle BCE$.

$\angle BEC = \beta + \alpha$ (from ②)

$\angle CBE = \beta + \alpha$ (from ①).

$\therefore \angle BEC = \angle CBE$.

$\triangle BCE$ is isosceles.

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3 (i) Given $f(x) = x^3 + ax + b$.

Given $f(-3) = 0$.

$-27 - 3a + b = 0$

$b = 3a + 27$ —— ①

Given $f(4) = 56$.

$64 + 4a + b = 56$.

$4a + b = -8$. —— ②.

Sub ① into ②.

$4a + 3a + 27 = -8$.

$7a = -35$.

$a = -5$.

Put $a = -5$ into ①

$b = 3(-5) + 27$

$b = 12$.

$\therefore a = -5,\ b = 12$ #.

(ii) $f(x) = x^3 - 5x + 12$

From (i) $x+3$ is a factor of $f(x)$.

$f(x) = (x+3)(x^2 - 3x + 4)$

$$
\begin{array}{r}
x^2 - 3x + 4 \\
x+3 \overline{) x^3 + 0x^2 - 5x + 12} \\
-)\ \underline{x^3 + 3x^2 \qquad\qquad} \\
-3x^2 - 5x \quad \\
-)\ \underline{-3x^2 - 9x \quad} \\
4x + 12 \\
-)\ \underline{4x + 12} \\
0
\end{array}
$$

For $f(x) = 0$, we have.

$(x+3)(x^2 - 3x + 4) = 0$.

$x + 3 = 0$. $\qquad x^2 - 3x + 4 = 0$.

$x = -3$. $\qquad$ Using $b^2 - 4ac$

$= (-3)^2 - 4(4)$

$= 9 - 16$

$= -7$

Since $b^2 - 4ac = -7 < 0$ for $x^2 - 3x + 4$ (no solution).

$\therefore$ Number of real roots of $f(x) = 0$, is 1 #

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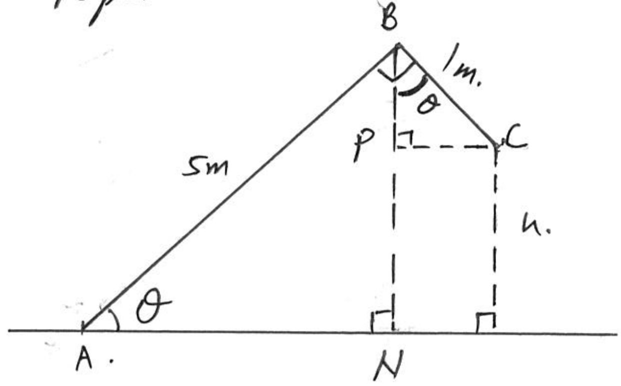

4(i) In ΔABN,

$$\sin\theta = \frac{BN}{AB.}$$

$$BN = 5\sin\theta \text{ . m}$$

Note : $\measuredangle ABN = 180^\circ - 90^\circ - \theta$ (Sum of $\measuredangle$s in Δ).

$$\measuredangle ABN = 90^\circ - \theta.$$

$$\therefore \measuredangle PBC = 90^\circ - \measuredangle ABN$$
$$= 90^\circ - (90^\circ - \theta)$$
$$= \theta$$

In ΔBPC,

$$\cos\theta = \frac{BP}{BC}$$
$$BP = 1 \times \cos\theta$$
$$BP = \theta \text{ m}$$

$$\therefore h = BN - BP$$
$$= (5\sin\theta - \cos\theta.) \text{ m.}$$

(ii) $h = 5\sin\theta - \cos\theta = R\sin(\theta - \alpha)$, wher $R > 0$ and $0^\circ < \alpha < 90^\circ$.

$$5\sin\theta - \cos\theta = R\cos\alpha\sin\theta - R\sin\alpha\cos\theta$$

By comparing,

$$R\cos\alpha = 5 \text{ ——— } (1)$$
$$R\sin\alpha = 1 \text{ ——— } (2).$$

$(1)^2 + (2)^2$, $R^2[\cos^2\alpha + \sin^2\alpha] = 5^2 + 1^2.$

$$R = \sqrt{26} \quad \text{(Given } R > 0 \text{).}$$

$(2)/(1)$ $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{5}.$

$$\tan\alpha = \frac{1}{5}.$$
$$\alpha = 11.309^\circ$$

$$\therefore h = \sqrt{26}\sin(\theta - 11.3^\circ)$$

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4 (iii). Given $h = 3\text{ m}$.

Then $\sqrt{26}\sin(\theta - 11.309^\circ) = 3$.

$$\sin(\theta - 11.309^\circ) = \frac{3}{\sqrt{26}}.$$

$$\text{basic } \angle = \sin^{-1}\left(\frac{3}{\sqrt{26}}\right)$$

$$= 36.0399^\circ$$

Since θ is acute, then

$$\theta - 11.309^\circ = 36.0399^\circ$$

$$\theta = 47.3487$$

$$\theta \approx 47.3^\circ \quad (\text{Corr to 1 dec pl}).$$

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5 (i).
$$3\cos^2 x - \sin^2 x$$
$$= 2\cos^2 x + \cos^2 x - \sin^2 x$$
$$= \cos 2x + 1 + \cos 2x$$
$$= 1 + 2\cos 2x \quad (a = 1,\ b = 2)$$

(ii)
$$\int (3\cos^2 x - \sin^2 x)$$
$$= \int (1 + 2\cos 2x)\, dx \quad \{\text{use result (i)}\}$$
$$= x + \sin 2x + C, \text{ where } C \text{ is a constant.}$$

$$\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} (3\cos^2 x - \sin^2 x)\, dx = \left[x + \sin 2x\right]_{-\frac{\pi}{12}}^{\frac{\pi}{12}}$$
$$= \frac{\pi}{12} + \sin\frac{\pi}{6} - \left(-\frac{\pi}{12}\right) - \sin\left(-\frac{\pi}{6}\right)$$
$$= \frac{\pi}{6} + 2\sin\frac{\pi}{6}$$
$$= \frac{\pi}{6} + 1$$
$$\approx 1.52 \quad (\text{corr to 3 sig fig})$$

6 (i). Given $y = p + q \sin 3x$, $p > 0$, $q > 0$.

(i) Period of $y = \frac{360°}{3} = 120°$.

(ii) amplitude of $y = \frac{6-(-2)}{2} = 4$.

(iii) q = amplitude of $y = 4$.

Given max of $y = 6$.

$p + 4 = 6$

$p = 2$

$\therefore p = 2,\ q = 4$.

(iv) $y = 2 + 4 \sin 3x$.

When $y = 0$.

$2 + 4\sin 3x = 0$.

$\sin 3x = -\frac{1}{2}$.

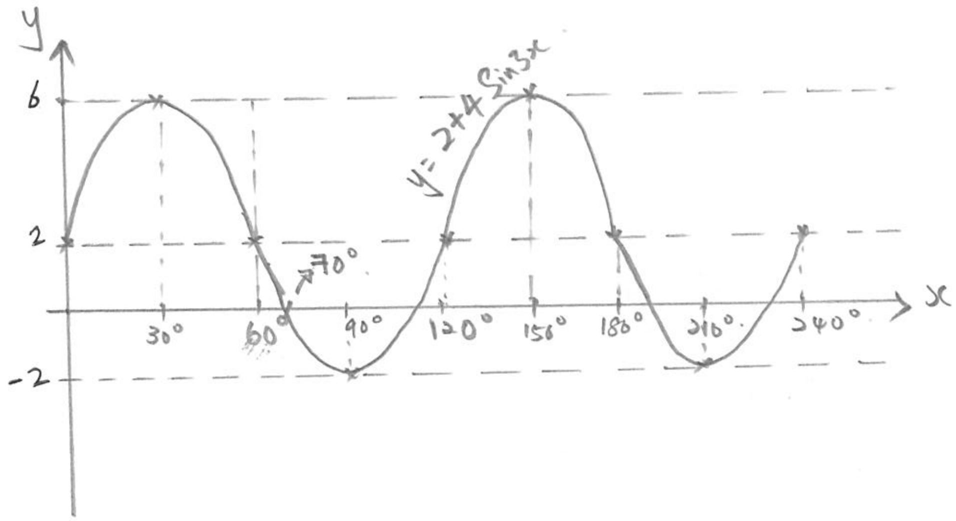
basic $\measuredangle = 30°$.

$\therefore 3x = 210°, 330°$,

$x = 70°, 110°$.

$\therefore$ smallest positive value of $x = 70°$.

(v).



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7. Given total length of the tape = 600cm.

$$2(2r) + 2l + 2(2\pi r) = 600$$
$$2r + l + 2\pi r = 300.$$
$$l = (300 - 2r - 2\pi r) \text{ cm.} \quad \text{—— (A)}$$

Volume of cylinder, $V = \pi r^2 h.$
$$V = \pi r^2 l.$$
$$= \pi r^2 (300 - 2r - 2\pi r) \text{ cm}^3. \text{ (shown).}$$

$$V = 300\pi r^2 - (2 + 2\pi)\pi r^3$$

$$\frac{dV}{dr} = 600\pi r - 6(1+\pi)\pi r^2.$$

For $\frac{dV}{dr} = 0$, $\quad 600\pi r - 6(1+\pi)\pi r^2 = 0.$
$$r[600\pi - 6(1+\pi)\pi r] = 0.$$

Since $r > 0$, then
$$600\pi - 6(1+\pi)\pi r = 0.$$
$$6(1+\pi)\pi r = 600\pi$$
$$(1+\pi) r = 100$$
$$r = \frac{100}{1+\pi} \text{ cm}$$

$\therefore k = 100.$

sub $r = \frac{100}{1+\pi}$ into (A).

$$l = 300 - 2(1+\pi)\frac{100}{1+\pi}$$
$$= 300 - 200$$
$$= 100.$$
$$l = 100 \text{ cm}$$

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8 (a)
$$\frac{\sqrt{6}+\sqrt{5}}{\sqrt{15}-\sqrt{2}} = \frac{(\sqrt{6}+\sqrt{5})(\sqrt{15}+\sqrt{2})}{(\sqrt{15}-\sqrt{2})(\sqrt{15}+\sqrt{2})}$$
$$= \frac{\sqrt{90}+\sqrt{12}+\sqrt{75}+\sqrt{10}}{15-2}$$
$$= \frac{3\sqrt{10}+2\sqrt{3}+5\sqrt{3}+\sqrt{10}}{13}$$
$$= \frac{4}{13}\sqrt{10} + \frac{7}{13}\sqrt{3}$$

$\therefore a = \frac{4}{13}, \ b = \frac{7}{13}$ #.

(b)
$$2^{2x} = 2^{2+x} + 21$$
$$(2^x)^2 = 4(2^x) + 21$$

$(2^x)^2 - 4(2^x) - 21 = 0$ #

$$(2^x)^2 - 4(2^x) - 21 = 0$$
$$(2^x - 7)(2^x + 3) = 0$$

$2^x - 7 = 0$ or $2^x + 3 = 0$

$2^x = 7$ $\qquad$ $2^x = -3$ (reject as $2^x > 0$)

$$x = \frac{\lg 7}{\lg 2}$$
$$= 2.807$$
$$x \approx 2.81 \text{ (Corr to 2 dec pl)}$$

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9 (i) Given $a = -16e^{-0.5t}$

$$\frac{dv}{dt} = -16e^{-0.5t}$$
$$v = \int -16e^{-0.5t}\,dt$$
$$v = 32e^{-0.5t} + C \quad \text{, where } C \text{ is a constant.}$$

Given $v = 28\ ms^{-1}$ when $t = 0$.

$$28 = 32 + C.$$
$$C = -4$$
$$\therefore v = 32e^{-0.5t} - 4 \quad \text{———— (1)}$$

When $v = 0$, $0 = 32e^{-0.5t} - 4$.
$$e^{-0.5t} = \frac{1}{8}.$$
$$-0.5t = \ln\frac{1}{8}.$$
$$0.5t = \ln 8$$
$$t = 2\ln 8$$
$$t = 6\ln 2.$$
$$\approx 4.16\ s \quad \text{((corr to 3 sig fig).}$$

From (1),
$$\frac{ds}{dt} = 32e^{-0.5t} - 4$$
$$s = \int(32e^{-0.5t} - 4)\,dt.$$
$$s = -64e^{-0.5t} - 4t + C_1 \quad \text{, where } C_1 \text{ is a constant}$$

$s = 0$ when $t = 0$.
$$0 = -64 - 0 + C_1$$
$$C_1 = 64.$$
$$\therefore s = -64e^{-0.5t} - 4t + 64.$$

when $t = 6\ln 2$ (at rest)

$$s = -64e^{-3\ln 2} - 24\ln 2 + 64.$$
$$= -8 - 24\ln 2 + 64$$
$$= 56 - 24\ln 2 \approx 39.4\,m \quad \text{(corr to 3 sig fig)}$$

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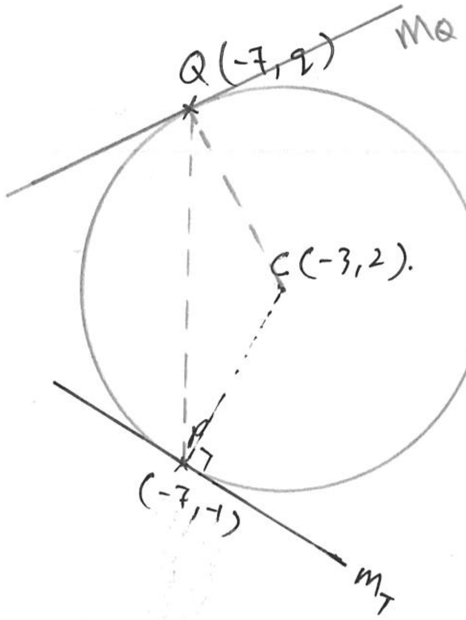
10 (i) $x^2 + y^2 + 6x - 4y - 12 = 0.$

$$x^2 + 6x + 3^2 + y^2 - 4y + 2^2 = 12 + 3^2 + 2^2.$$
$$(x+3)^2 + (y-2)^2 = 25.$$
$$(x+3)^2 + (y-2)^2 = 5^2. \quad \text{——— } (1)$$

Center, $C = (-3, 2)$.
radius $= 5$ units



(ii) $m_{CP} = \frac{2+1}{-3+7} = \frac{3}{4}.$

$$m_T = -\frac{1}{m_{CP}}$$
$$= -\frac{4}{3}.$$

Equation of the tangent:

$$y = -\frac{4}{3}x + c.$$

$y = -1$ when $x = -7$

$$-1 = -\frac{4}{3}(-7) + c.$$
$$c = -\frac{31}{3}$$

$$\therefore \quad y = -\frac{4}{3}x - \frac{31}{3}.$$
$$3y = -4x - 31$$
$$3y + 4x + 31 = 0 \ \# . \quad \text{(shown)}. \quad \text{——— } (2)$$

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10 (iii) Let $Q(-7, q)$, $\because$ same distance from the y-axis as point P

Since Q lies on circle, using ①, we have.

$$(-7+3)^2 + (y-2)^2 = 25.$$
$$(y-2)^2 = 25 - 16.$$
$$y - 2 = 3 \quad \text{or} \quad y - 2 = -3.$$
$$y = 5 \qquad\qquad y = -1.$$

(reject $\because$ the value belongs to point P)

$\therefore Q(-7, 5)$.

$$m_{QC} = \frac{5-2}{-7+3} = -\frac{3}{4}.$$
$$\text{Gradient}, \ m_Q = -\frac{1}{m_{QC}} = \frac{4}{3}.$$

Equation of tangent at Q:
$$y = \frac{4}{3}x + c$$

$y = 5$, when $x = -7$.
$$5 = \frac{4}{3}(-7) + c.$$
$$c = \frac{43}{3}.$$
$$\therefore y = \frac{4}{3}x + \frac{43}{3}.$$
$$3y = 4x + 43 \ \# \quad \text{——— ③}.$$

(iv) Sub ③ into ②.
$$4x + 43 + 4x + 31 = 0.$$
$$8x = -74$$
$$x = -\frac{74}{8}$$
$$x = -\frac{37}{4}$$

Put $x = -\frac{37}{4}$ into ③, $3y = -37 + 43.$
$$y = 2.$$

$$\therefore R\left(-\frac{37}{4}, 2\right) \ \#$$

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11 Given $y = 3 - \frac{12}{(x+3)^2}$

when $x = 0$, $y = 3 - \frac{12}{3^2} = \frac{5}{3}$.

$\therefore A(0, \frac{5}{3})$.

when $y = 0$, $0 = 3 - \frac{12}{(x+3)^2}$

$(x+3)^2 = 4$.

$x + 3 = 2$ or $x + 3 = -2$.

$x = -1$ $\quad$ $x = -5$.

$\therefore B(-1, 0)$, $C(-5, 0)$.

Given AD is parallel to x-axis.

$\therefore$ y-coordinates of D = y-coordinates of A.

$= \frac{5}{3}$.

when $y = \frac{5}{3}$,

$\frac{5}{3} = 3 - \frac{12}{(x+3)^2}$

$\frac{12}{(x+3)^2} = \frac{4}{3}$.

$(x+3)^2 = 9$

$x + 3 = 3$ or $x + 3 = -3$.

$x = 0$ $\quad$ $x = -6$

(reject $\because$ the value belongs to point A)

$\therefore$ Coordinates of D $= (-6, \frac{5}{3})$ #.

$A(0, \frac{5}{3})$, $B(-1, 0)$, $C(-5, 0)$, $D(-6, \frac{5}{3})$.

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1(ii) Area bounded by the curve AB and the coordinates axes

$= \int_{-1}^{0} 3 - \frac{12}{(x+3)^2}\, dx$

$= \left[3x + \frac{12}{x+3} \right]_{-1}^{0}$

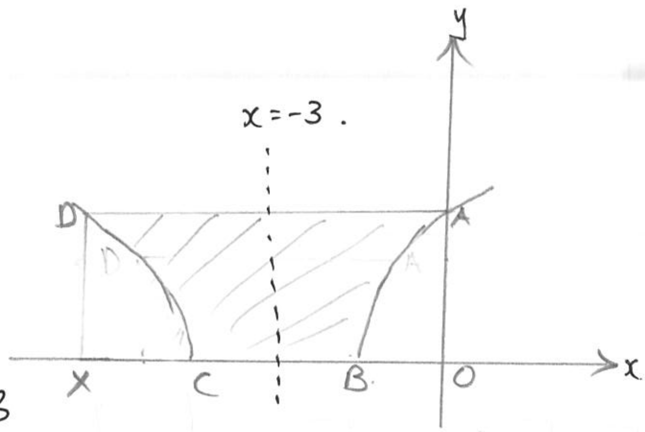
$= 0 + \frac{12}{3} - (-3) - \frac{12}{2}$

$= 1$ unit^2.

(iii). Area of the rect OADX

$= 6 \times \frac{5}{3}$

$= 10$ units^2.



Area of CDX = Area of OAB

= 1 unit^2. (curve CD is symmetrical to curve BA about $x = -3$.)

∴ Area of shaded region $= 10 - 1 - 1$

$= 8$ units^2.