ELEMENTARY MATHEMATICS 4016/02

Paper 2 Suggested Solutions October/November 2011

1. **Topic: Algebra (Formulae, Algebraic Manipulation)**

(a)
$$\begin{aligned}
\frac{2x+7}{3} &= \frac{3-x}{5}\\
5(2x+7) &= 3(3-x)\\
10x+35 &= 9-3x\\
10x+3x &= 9-35\\
13x &= -26\\
\boldsymbol{x} &= \mathbf{-2}
\end{aligned}$$

(b)
$$\begin{aligned}
xy &= 2(x+3)\\
xy &= 2x+6\\
xy-2x &= 6\\
x(y-2) &= 6\\
\boldsymbol{x} &= \boldsymbol{\frac{6}{y-2}}
\end{aligned}$$

Group all variables containing x.

(c)
$$\begin{aligned}
\frac{4}{x-2}+\frac{2}{2x+1} &= \frac{4(2x+1)+2(x-2)}{(x-2)(2x+1)}\\
&= \frac{8x+4+2x-4}{(x-2)(2x+1)}\\
&= \boldsymbol{\frac{10x}{(x-2)(2x+1)}}
\end{aligned}$$

(d)
$$\begin{aligned}
\frac{4x^2-y^2}{2x^2+xy} &= \frac{(2x+y)(2x-y)}{2x^2+xy}\\
&= \frac{(2x+y)(2x-y)}{x(2x+y)}\\
&= \boldsymbol{\frac{2x-y}{x}}
\end{aligned}$$

$a^2-b^2=(a+b)(a-b)$

2. **Topic: Arithmetic (Application of Mathematics in Practical Situations & Compound Interest)**

(a) (i)
$$\begin{aligned}
250\text{ km} &\rightarrow 15.75\text{ litres}\\
1\text{ km} &\rightarrow \frac{15.75}{250} = 0.063\text{ litres}\\
100\text{ km} &\rightarrow 0.063\times 100 = 6.3\text{ litres}
\end{aligned}$$

Fuel consumption of the car is 6.3 litres per 100 km.

(ii) (a)
$$\begin{aligned}
8.2\text{ litres} &\rightarrow 100\text{ km}\\
1\text{ litre} &\rightarrow \frac{100}{8.2} = 12.195\text{ km}\\
60\text{ litres} &\rightarrow 12.195\times 60 = 731.7\\
&\approx 732\text{ km } (\textbf{(3 sig. fig.)}
\end{aligned}$$

(b)
$$\begin{aligned}
100\text{ km} &\rightarrow 8.2\text{ litres}\\
1\text{ km} &\rightarrow \frac{8.2}{100} = 0.082\text{ litres}\\
120\text{ km} &\rightarrow 0.082\times 120 = 9.84\text{ litres}\\
\text{Cost of petrol} &= 9.84\times \$1.65\\
&= \$16.236\\
&\approx \mathbf{\$16.24\ (nearest\ cent)}
\end{aligned}$$

(b) (i)
$$\begin{aligned}
5\text{ units} &\rightarrow \$1000\\
1\text{ unit} &\rightarrow \frac{1000}{5} = \$200\\
12\text{ units} &\rightarrow 200\times 12\\
&= \mathbf{\$2400}
\end{aligned}$$

(ii)
$$\begin{aligned}
\text{Total amount of money} &= P\left(1+\frac{r}{100}\right)^n\\
&= \$1000\left(1+\frac{3.5}{100}\right)^5\\
&= \$1187.6863\\
&\approx \mathbf{\$1187.69\ (nearest\ cents)}
\end{aligned}$$

3. **Topic: Angle Properties of Polygon & Geometry (Geometrical Properties of Circles)**

(a) Ext $\measuredangle$ = 180° – 162°
= 18°

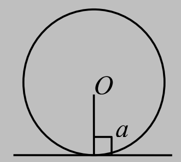
Number of sides = $\frac{360°}{18}$
= **20**

Sum of exterior angles of polygon is 360° ⇒ Number of sides = $\frac{360°}{\text{ext.}\measuredangle}$

(b) (i) (a) $O\hat{A}P$ = 90° ($OA \perp AP$)

$A\hat{O}P = 180° - O\hat{A}P - A\hat{P}O$
= 180° − 90° − 36° (Sum of $\measuredangle$ in Δ)
= **54°**

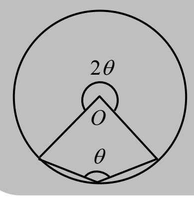
Radius ⊥ Tangent
∠a = 90°



(b) $C\hat{O}B = A\hat{O}P$ = 54°

$O\hat{B}C = \frac{180° - C\hat{O}B}{2}$ (isos. Δ, OC = OB)
$= \frac{180° - 54°}{2}$
= **63°**

∠ at centre = 2 ×∠ at circumference

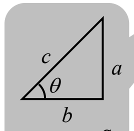


(c) $\angle ACB = \frac{\text{reflex } A\hat{O}B}{2}$
$= \frac{360° - 54(2)}{2}$ (∠at centre = 2∠at circumference)
= **126°**

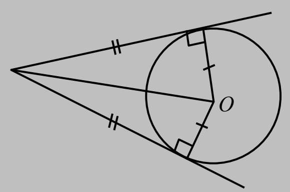
(ii) tan 36° = $\frac{6}{AP}$

AP = 8.25829

Area of quad $AOBP$ = $2 \times \frac{1}{2} \times AP \times AO$
= 8.25829 × 6
= 49.459
≈ **49.5 cm^2 (3 sig.fig.)**

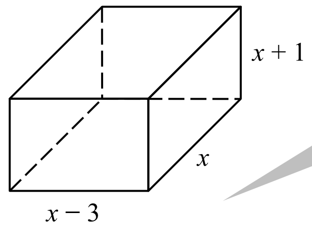


$\tan\theta = \frac{a}{b}$



4. **Topic: Solutions to Quadratic Equations**

(a) Length of the block = x cm
Height of the block = $(x + 1)$ cm
Width of the block = $(x – 3)$ cm



Total Surface area =
2(length × width) + 2(width × height) + 2(length × height)

Total surface area = $2x(x – 3) + 2(x – 3)(x + 1) + 2x(x + 1)$
= $2x^2 – 6x + 2(x^2 – 2x – 3) + 2x^2 + 2x$
= $2x^2 – 6x + 2x^2 – 4x – 6 + 2x^2 + 2x$
= **$6x^2 – 8x – 6$ cm^2**

(b) $6x^2 – 8x – 6 = 500$
$6x^2 – 8x – 506 = 0$
$3x^2 – 4x – 253 = 0$ (shown)

General quadratic formula
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(c) $3x^2 – 4x – 253 = 0$

$x = \frac{4 \pm \sqrt{4^2 - 4(3)(-253)}}{2(3)}$
$= \frac{4 \pm \sqrt{3052}}{6}$
= 9.874 or −8.5408
≈ **9.87 or −8.54 (2 d.p.)**

Question simply asks to solve the equation. Do NOT reject the negative value of x here!

(d) Height of the block $= x + 1$

$= 9.874 + 1$

$= 10.874$

$=$ **10.9 cm (3 sig.fig.)**

5. **Topic: Number Patterns & Algebra**

(a) (i)

$$T_n = \frac{n(n+1)}{4}$$

$$T_{20} = \frac{20(21)}{4}$$

$$= \mathbf{105}$$

(ii)

$$33 = \frac{n(n+1)}{4}$$

$$132 = n^2 + n$$

$$n^2 + n - 132 = 0$$

$$(n + 12)(n - 11) = 0$$

$n + 12 = 0$ or $n - 11 = 0$

$n = -12$ (rej) or $n = 11$

∴ **The term is $\mathbf{T_{11}}$.**

(b) (i) $\frac{2p+1}{2} = p + \frac{1}{2}$

(2p + 1) is not divisible by 2 since p is an integer.

∴ **2p + 1 is an odd number.**

(ii) Next odd number $= (2p + 1) + 2$

$= \mathbf{2p + 3}$

(iii) $(2p + 1)^2 = 4p^2 + 4p + 1$

$\mathbf{(2p + 3)^2 = 4p^2 + 12p + 9}$

(iv) $(2p + 3)^2 - (2p + 1)^2$

$= 4p^2 + 12p + 9 - (4p^2 + 4p + 1)$

$= 8p + 8$

$= \mathbf{8(p + 1)}$

Since p is an integer, 8(p + 1) is always a multiple of 8.

6. **Topic: Trigonometry (Cosine Rule, Sine Rule, Bearings, Area of triangle, Angle of Elevation)**

(a) (i)

$$LB^2 = 250^2 + 400^2 - 2(250)(400)\cos 65°$$

$$LB = 371.45$$

$$\approx \mathbf{371\ m}$$

Cosine rule (given in formula sheet): $a^2 = b^2 + c^2 - 2bc\cos A$

(ii) Area of $\Delta LAB = \frac{1}{2}(250)(400)\sin 65°$

$= 45315.38$

$\approx$ **45300 m^2 (3 sig. fig.)**

Given in formula sheet: Area of $\Delta = \frac{1}{2}ab\sin C$

(iii)

$$\frac{LA}{\sin L\hat{B}A} = \frac{LB}{\sin L\hat{B}A}$$

$$\frac{250}{\sin L\hat{B}A} = \frac{371.45}{\sin 65°}$$

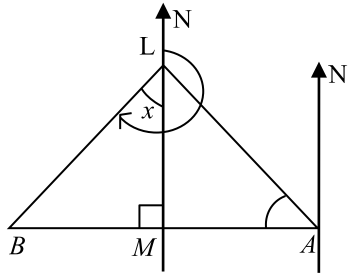
$$\sin L\hat{B}A = 0.60997$$

$$L\hat{B}A = 37.588$$

$$\approx \mathbf{37.6° \ (1d.p)}$$

Sine rule (given in formula sheet): $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

(iv)



Since B is due west of A, i.e. $L\hat{M}B = 90°$

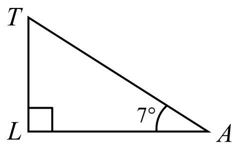
$$\begin{aligned} \angle x &= 180° - 90° - L\hat{B}A \\ &= 180° - 90° - 37.588° \\ &= 52.412° \end{aligned}$$

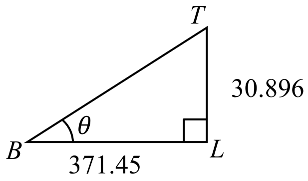
$$\begin{aligned} \therefore \text{Bearing of } B \text{ from } L &= 180° + x° \\ &= 180° + 52.412 \\ &\approx \mathbf{232.4° \ (1\ d.\ p.)} \end{aligned}$$

Rounded off to one decimal place since bearing is in degrees.

(b) Let TL be the height of the lighthouse.

$$\begin{aligned} \tan 7° &= \frac{TL}{250} \\ TL &= 30.696 \text{ m} \end{aligned}$$





Let θ be the angle of elevation of the top of the lighthouse from B.

$$\begin{aligned} \tan\theta &= \frac{30.696}{371.45} \\ \theta &= 4.724° \\ &= \mathbf{4.7° \ (1\ d.\ p.)} \end{aligned}$$

7. **Topic: Variation**

(a) Let V be the quantity of paint and x be the depth of the container.

$$V = kx^2$$

When $x = 50$, $V = 150$,

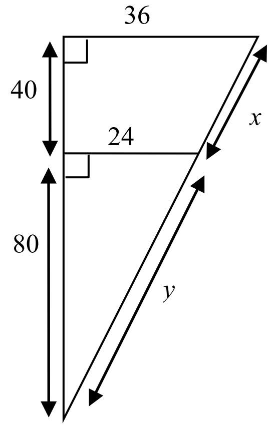
$$\begin{aligned} 150 &= k\,(50)^2 \\ k &= 0.06 \\ V &= 0.06x^2 \end{aligned}$$

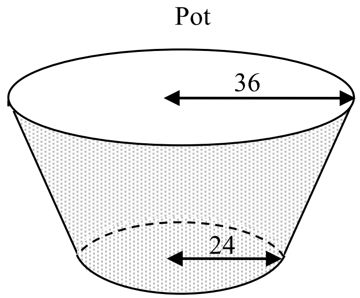
(i) When $x = 70$ cm, $V = 0.06(70)^2$
$= \mathbf{294\ ml}$

(ii) When $V = 54$ ml, $54 = 0.06(x)^2$

$$\begin{aligned} x^2 &= 900 \\ \boldsymbol{x} &= \mathbf{30\ cm} \text{ or } -30 \text{ (rej)} \end{aligned}$$

(b)

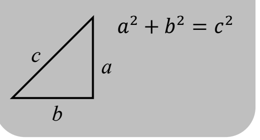




(i)

$$y^2 = 80^2 + 24^2$$
$$y = \sqrt{6976}$$
$$(x+y)^2 = 120^2 + 36^2$$
$$x = \sqrt{15696} - \sqrt{6976}$$
$$= 41.76$$
$$\approx \textbf{41.8 m (3 sig.fig.) (shown)}$$

Pythagoras Theorem



(ii) Surface area of the pot

$$= \underbrace{\pi(24)^2}_{\text{Base area}} + \underbrace{\pi(36)(x+y) - \pi(24)(y)}_{\text{Curved surface area}}$$
$$= \pi(24)^2 + \pi(36)\sqrt{15696} - \pi(24)\sqrt{6976}$$
$$= 9681.36$$
$$\approx \textbf{9680 cm}^2 \textbf{ (3 sig. fig.)}$$

"Total surface area of the outside of the pot"
= curved surface area + base area

(iii)

$$\frac{\text{Volume of the smaller pot}}{\text{Volume of the larger pot}} = \left(\frac{\text{Height of smaller pot}}{\text{Height of larger pot}}\right)^3$$
$$\frac{1}{2} = \left(\frac{40}{\text{Height of larger pot}}\right)^3$$
$$\sqrt[3]{\frac{1}{2}} = \frac{40}{\text{Height of larger pot}}$$
$$\text{Height of larger pot} = \frac{40}{\sqrt[3]{\frac{1}{2}}}$$
$$= 50.396$$
$$\approx \textbf{50.4 cm (3 sig.fig.)}$$

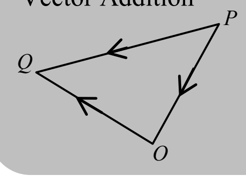
Volume of similar figures
$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$

8. **Topic: Vectors in Two Dimensions**

(a) (i) $\overrightarrow{OP} = \begin{pmatrix}3\\4\end{pmatrix}$ $\overrightarrow{OQ} = \begin{pmatrix}-1\\2\end{pmatrix}$

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= \begin{pmatrix}-3\\-4\end{pmatrix} + \begin{pmatrix}-1\\2\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{-4}\\\mathbf{-2}\end{pmatrix}$$

Triangle Law of Vector Addition



(ii) $$|\overrightarrow{PQ}| = \sqrt{(-4)^2 + (-2)^2}$$
$$= 4.472$$
$$\approx \textbf{4.47 units (3 sig.fig.)}$$

Magnitude of $\begin{pmatrix}u\\v\end{pmatrix}$:
$\left|\begin{pmatrix}u\\v\end{pmatrix}\right| = \sqrt{u^2 + v^2}$

(iii) $$\overrightarrow{PL} = \frac{1}{2}\overrightarrow{PQ}$$
$$= \frac{1}{2}\begin{pmatrix}-4\\-2\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{-2}\\\mathbf{-1}\end{pmatrix}$$

(b) (i) (a) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\mathbf{a} + \mathbf{b}$

(b) $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$\overrightarrow{AB} = 2\overrightarrow{AX}$

$= \mathbf{a} + \frac{1}{2}\overrightarrow{AB}$

$= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b})$

$= \mathbf{a} - \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

(c) $\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD}$

$\overrightarrow{OB} = 2\overrightarrow{BC} \Rightarrow \overrightarrow{OC} = \frac{1}{2}\overrightarrow{BO} + \overrightarrow{BO}$

$\overrightarrow{AD} = 2\overrightarrow{OA} \Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + 2\overrightarrow{OA}$

$= -\frac{3}{2}\mathbf{b} + 3\mathbf{a}$

$= 3\mathbf{a} - \frac{3}{2}\mathbf{b}$

(ii) $\overrightarrow{OY} = \overrightarrow{OC} + \overrightarrow{CY}$

$CY : YD = 1 : 2$

$= \frac{3}{2}\mathbf{b} + \frac{1}{3}\overrightarrow{CD}$

$= \frac{3}{2}\mathbf{b} + \frac{1}{3}\left[-\frac{3}{2}\mathbf{b} + 3\mathbf{a}\right]$

$= \frac{3}{2}\mathrm{b} - \frac{1}{2}\mathrm{b} + \mathrm{a}$

$= \mathbf{a} + \mathbf{b}$

$\overrightarrow{OX} = k\overrightarrow{OY}$
$\Rightarrow O, X, Y$ are collinear (straight line)

(iii) $\overrightarrow{OX} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

$= \frac{1}{2}\overrightarrow{OY}$

⇒ *O*, *X* and *Y* are collinear
⇒ *X* is the midpoint of line *OY*.

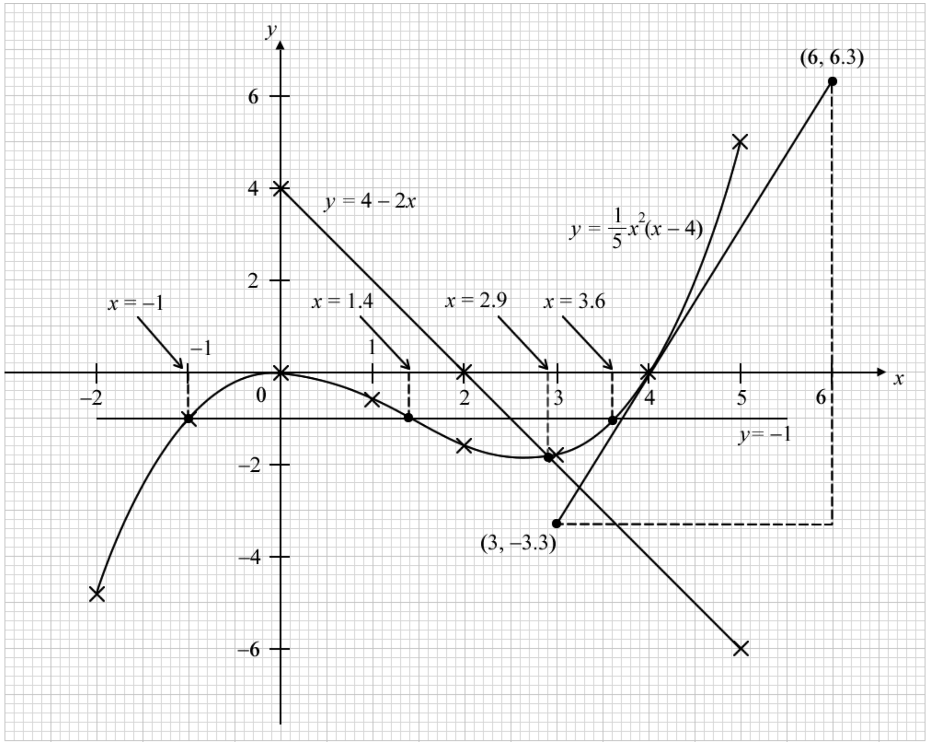
9. **Topic: Graphical Solution of Equations**

(a) $y = \frac{1}{5}x^2(x-4)$

When $x = -2$, $y = \frac{1}{5}(-2)^2(-2-4)$

$= \mathbf{-4.8}$

(b)



(c) (i) $\frac{1}{5}x^2(x-4) = -1$

$y = -1$

∴ x = −1, 1.4 or 3.6

(d) Gradient $= \frac{6.3-(-3.3)}{6-3}$

$=$ **3.2**

AMaths students:

Check: $\frac{dy}{dx} = \frac{1}{5}[2x(x-4)+x^2]$

Sub $x = 4 \Rightarrow \frac{dy}{dx} = 3.2$

(e) (ii) x = **2.9**

(iii) $y = 4 - 2x$ ………………………………… (1)

$y = \frac{1}{5}x^2(x-4)$………………………………… (2)

(1) = (2), $4 - 2x = \frac{1}{5}x^2(x-4)$

$5(4-2x) = x^3 - 4x^2$

$20 - 10x = x^3 - 4x^2$

$x^3 - 4x^2 + 10x - 20 = 0$

∴ A = 10, B = −20

10. **Topics: Statistics, Simple Probability**

(a) (i) Total number of students = 100

% of students who received less than 20 emails in a week

$= \frac{21}{100} \times 100\%$

= **21%**

$\frac{\text{no. of students who received less than 20 emails in a week}}{\text{total no. of students}} \times 100\%$

(ii) (a) Mean number of emails received in a week

$= \frac{\sum fx}{\sum f}$

$= \frac{8(5)+13(15)+25(25)+30(35)+18(45)+6(55)}{100}$

$=$ **30.5**

(b) Standard deviation

$= \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

$= \sqrt{\frac{8(5)+13(15)+25(25)+30(35)+18(45)+6(55)}{100} - (30.5)^2}$

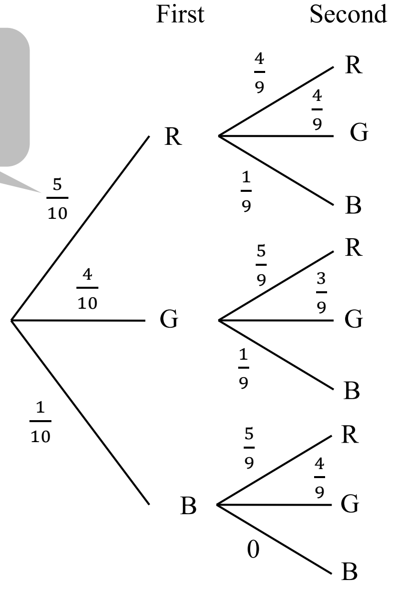
$= 13.067$

$\approx$ **13.1 (3 sig. fig.)**

$\frac{\sum fx}{\sum f} = 30.5$ from (ii)(a)

(b) (i)

Check: Sum of probabilities of branches at each level equals one



(ii) (a) P (two blue sweets are taken) = 0

(b) P (both sweets are of the same colour)

$= \frac{5}{10}\left(\frac{4}{9}\right) + \frac{4}{10}\left(\frac{3}{9}\right)$

$= \mathbf{\frac{16}{45}}$

P(RR) + P(GG) + P(BB)
P(BB) = 0

(c) P(one of the sweets taken is blue)

$= 1 - $ P (none of the sweets taken is blue)

$= 1 - \frac{9}{10}\left(\frac{8}{9}\right)$

$= \mathbf{\frac{1}{5}}$

P(none of the sweets taken is blue)
$=$ P(1st sweet not blue) ∩ P(2nd sweet not blue)
$= \frac{9}{10}\left(\frac{8}{9}\right)$