ADDITIONAL MATHEMATICS — 4038/02

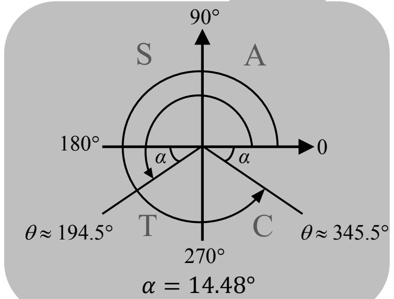
Paper 2 Suggested Solutions — **October/November 2010**

1. **Topic: Trigonometric Functions**

$$3\cot^2\theta + 10\operatorname{cosec}\theta = 5,\ 0° \le \theta \le 360°$$
$$3(\operatorname{cosec}^2\theta - 1) + 10\operatorname{cosec}\theta = 5$$
$$3\operatorname{cosec}^2\theta + 10\operatorname{cosec}\theta - 8 = 0$$
$$(3\operatorname{cosec}\theta - 2)(\operatorname{cosec}\theta + 4) = 0$$
$$3\operatorname{cosec}\theta - 2 = 0 \text{ or } \operatorname{cosec}\theta + 4 = 0$$
$$\operatorname{cosec}\theta = \tfrac{2}{3} \qquad \operatorname{cosec}\theta = -4$$
$$\sin\theta = \tfrac{3}{2}\text{ (reject)} \qquad \sin\theta = -\tfrac{1}{4}$$
$$\text{Basic } \angle\alpha = 14.48°$$
$$\therefore \theta = 180° + 14.48°,\ 360° - 14.48°$$
$$= 197.47°,\ 345.52°$$
$$\approx \mathbf{194.5°,\ 345.5°\ (1\ d.p.)}$$

Given in formula sheet: $\operatorname{cosec}^2 A = 1 + \cot^2 A$

$\operatorname{cosec} x = \dfrac{1}{\sin x}$



2. **Topic: Applications of Differentiation (Maxima & Minima)**

(i) For ΔRVU and ΔRPQ,

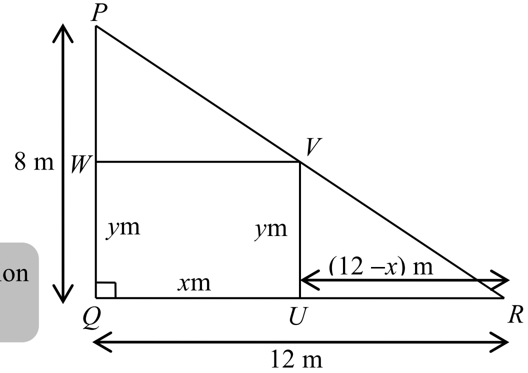
$\angle VRU = \angle PRQ$ (common)

$\angle UVR = \angle QPR$ (corresponding $\angle$s, VU // WQ since $QUVW$ is a rectangle)

$\Rightarrow \Delta RVU$ is similar to ΔRPQ.

Hence $\dfrac{RU}{RQ} = \dfrac{VU}{PQ}$

$$\frac{12-x}{12} = \frac{y}{8}$$
$$y = \frac{8}{12}(12-x)$$
$$\therefore y = 8 - \frac{2}{3}x \textbf{ (Shown)}$$



(ii)
$$A = xy$$
$$= x\left(8 - \frac{2}{3}x\right)$$
$$= \mathbf{8x - \frac{2}{3}x^2}$$

Sub expression of y from (i)

(iii)
$$\frac{dA}{dx} = 8 - \frac{4}{3}x$$

For a maximum value of A, $\dfrac{dA}{dx} = 0$,

$$8 - \frac{4}{3}x = 0$$
$$x = 6$$
$$\frac{d^2A}{dx^2} = -\frac{4}{3} < 0 \Rightarrow A \text{ is maximum when } x = 6 \text{ m}$$

When $x = 6$ m,

$$A = 8(6) - \frac{2}{3}(6)^2$$
$$= 24 \text{ m}^2$$

$\therefore$ **Maximum value of A is 24 m^2.**

2nd derivative test for stationary points of curve $y = f(x)$:
Max. point: $\dfrac{dy}{dx} = 0, \dfrac{d^2y}{dx^2} < 0$
Min. point: $\dfrac{dy}{dx} = 0, \dfrac{d^2y}{dx^2} > 0$

3. **Topic: Indices, Simultaneous Equations**

$$32^x \times 2^y = 1$$
$$(2^5)^x \times 2^y = 1$$
$$2^{5x} \times 2^y = 2^0$$

> $(a^m)^n = a^{mn}$

> $a^0 = 1$

$$5x + y = 0$$
$$y = -5x \quad -(1)$$
$$3^{x-12} \div 27^y = 81^{\frac{1}{x}}$$
$$3^{x-12} \div (3^3)^y = (3^4)^{\frac{1}{x}}$$
$$3^{x-12-3y} = 3^{\frac{4}{x}}$$

> $a^m \div a^n = a^{m-n}$

$$x - 12 - 3y = \frac{4}{x}$$
$$x^2 - 12x - 3xy = 4 \quad -(2)$$

Sub. (1) into (2),

$$x^2 - 12x - 3x(-5x) = 4$$
$$x^2 - 12x + 15x^2 = 4$$
$$16x^2 - 12x - 4 = 0$$
$$4x^2 - 3x - 1 = 0$$
$$(4x + 1)(x - 1) = 0$$
$$x = -\frac{1}{4} \quad \text{or} \quad x = 1$$

Sub. $x = -\frac{1}{4}$ into (1),

$$y = -5\left(-\frac{1}{4}\right) = \frac{5}{4}$$

Sub. $x = 1$ into (1),

$$y = -5(1) = -5$$

∴ The solutions are $x = -\frac{1}{4}$, $y = \frac{5}{4}$ and $x = 1$, $y = -5$.

4. **Topic: Binomial Expansions**

(i)
$$(r+1)^{\text{th}} \text{ term of } \left(x - \frac{k}{x^3}\right)^8 = \binom{8}{r} x^{8-r}(-kx^{-3})^r$$
$$= \binom{8}{r} x^{8-r}(-k)^r x^{-3r}$$
$$= \binom{8}{r} (-k)^r x^{8-4r} \quad -(1)$$

> In expanding $(a+b)^n$, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

In the constant term, $x^{8-4r} = x^0$

> Constant term = term that contains x^0

$$8 - 4r = 0$$
$$r = 2$$

Given that the constant term of $\left(x - \frac{k}{x^3}\right)^8 = 7$, sub $r = 2$ into (1),

$$\binom{8}{2}(-k)^2 = 7$$
$$28k^2 = 7$$
$$k^2 = \frac{1}{4}$$
$$k = \pm\frac{1}{2}$$

Since k is a positive constant, $\boldsymbol{k = \frac{1}{2}}$.

(ii) For $k = \frac{1}{2}$,

> Constant term in $\left(x - \frac{1}{2x^3}\right)^8 = 7$

$$(1 + x^4)\left(x - \frac{1}{2x^3}\right)^8 = (1 + x^4)\left[\ldots + 7 + \binom{8}{3}\left(-\frac{1}{2}\right)^3 x^{-4} + \cdots\right]$$
$$= (1 + x^4)[\ldots + 7 - 7x^{-4} + \ldots]$$

> x^{-4} term in $\left(x - \frac{1}{2x^3}\right)^8$
> $= \binom{8}{3}(x^{8-3})\left(-\frac{1}{2x^3}\right)^3$
> $= \binom{8}{3}\left(-\frac{1}{2}\right)^3 x^{-4}$

$$\text{Constant term} = (1)(7) + (x^4)(-7x^{-4})$$
$$= 7 - 7$$
$$= 0$$

∴ Since the constant term is zero, there is no constant term in the expansion. (Shown)

5. **Topic: Exponential & Logarithmic functions**

$y = 5 - e^{2x}$

When $x = 0$, $y = 5 - e^0 = 4$

> Curve intersects y-axis when $x = 0$

When $y = 0$, $0 = 5 - e^{2x}$

> Curve intersects x-axis when $y = 0$

$e^{2x} = 5$

$2x = \ln 5$

$x = \frac{1}{2}\ln 5$

$\Rightarrow A\ (0, 4)$ and $B\left(\frac{1}{2}\ln 5\ , 0\right)$

(i) Gradient of $AB = \frac{4-0}{0-\frac{1}{2}\ln 5} = -\frac{8}{\ln 5}$

> For points $A(x_1, y_1)$&$B(x_2, y_2)$, gradient of $AB = \frac{y_1 - y_2}{x_1 - x_2}$

Equation of line AB:

$y = -\frac{8}{\ln 5}x + 4$

> $A(0, 4) \Rightarrow y$-intercept is 4

Sub $x = \ln 5$, $y = k$,

$\therefore k = -\frac{8}{\ln 5}(\ln 5) + 4$

$= -8 + 4$

$= \mathbf{-4}$

> **Alternative Method:**
> Let P denote the point $(\ln 5, k)$.
> Since P lies on AB
> $\Rightarrow$ gradient of AP = gradient of BP
> $\frac{k-4}{\ln 5 - 0} = \frac{0-k}{\frac{1}{2}\ln 5 - \ln 5}$
> $\frac{k-4}{\ln 5} = \frac{-k}{-\frac{1}{2}\ln 5}$
> $k - 4 = 2k$

(ii) $x = \ln\sqrt{9-x}$

$x = \ln(9-x)^{\frac{1}{2}}$

$x = \frac{1}{2}\ln(9-x)$

> $\ln(x)^r = r\ln x$

$2x = \ln(9 - x)$

$e^{2x} = 9 - x$

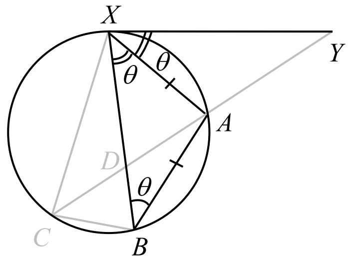
$-e^{2x} = x - 9$

$5 - e^{2x} = x - 4$

$y = x - 4$

$\therefore$ **Equation of the required line: $y = x - 4$**

6. **Topic: Geometric Proofs**



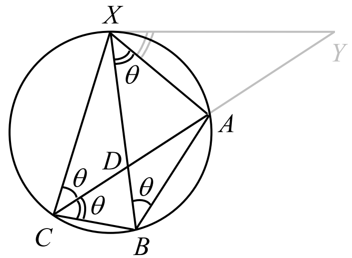
(i) $\angle AXB = \angle AXY = \theta$ (XA bisects $\angle YXB$)

$\angle ABX = \angle AXY = \theta$
(Alternate segment theorem)

$\therefore \angle AXB = \angle ABX = \theta$

$\Rightarrow AXB$ is isosceles.

Hence $AX = XB$ (Proved)



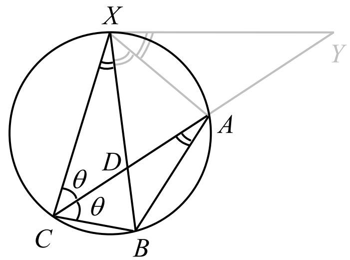
(ii) $\angle ACB = \angle AXB = \theta$ ($\angle$s in the same segment)

$\angle ACX = \angle ABX = \theta$ ($\angle$s in the same segment)

$\therefore \angle ACB = \angle ACX = \theta$

Since D lies on AC, CD bisects $\angle XCB$. (Proved)

> From (i),
> $\angle AXB = \angle ABX = \theta$

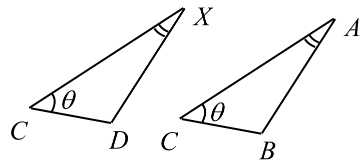


(iii) In ΔCDX and ΔCBA,

$\angle XCD = \angle ACB = \theta$ (Proven in (i))

$\angle CXD = \angle CAB$ ($\angle$s in the same segment)

$\therefore \Delta CDX$ and ΔCBA are similar. (Proved)



7. **Topic: Applications of Differentiation (Gradients, Tangents & Normals) & Integration (Area of aRegion)**

(i) $y = \sqrt{2x+5}$ – (1)

$\frac{dy}{dx} = \frac{1}{2}(2x+5)^{-\frac{1}{2}}(2)$

$= \frac{1}{\sqrt{2x+5}}$

$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$

Gradient of tangent at P, sub $x = 2$:

$\frac{dy}{dx} = \frac{1}{\sqrt{2(2)+5}} = \frac{1}{3}$

⇒ Gradient of $QR = \frac{1}{3}$ (since QR is parallel to the tangent to the curve at P)

At Q, sub $y = 0$ into (1):

$0 = \sqrt{2x+5}$

$2x + 5 = 0$

$x = -\frac{5}{2}$

$\Rightarrow Q\left(-\frac{5}{2},\ 0\right)$

Equation of straight line passing through (x_1, y_1): $\frac{y-y_1}{x-x_1} = \text{gradient } m$

∴ Equation of QR: $y - 0 = \frac{1}{3}\left[x - \left(-\frac{5}{2}\right)\right]$

$\boldsymbol{y = \frac{1}{3}x + \frac{5}{6}}$ – (2)

(ii) At R, sub $x = 2$ into (2):

$y = \frac{1}{3}(2) + \frac{5}{6} = \frac{3}{2}$

$\Rightarrow R\left(2, \frac{3}{2}\right)$

Length of ST = Length of RS = $\frac{3}{2}$ units

Length of QS = $2 - \left(-\frac{5}{2}\right)$ = $\frac{9}{2}$ units

Area of $\Delta QST = \frac{1}{2}(QS)(ST)$

$= \frac{1}{2}\left(\frac{9}{2}\right)\left(\frac{3}{2}\right)$

$= \frac{27}{8}$ units^2

$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

Area of region $QPS = \int_{-\frac{5}{2}}^{2} y \, dx$

$= \int_{-\frac{5}{2}}^{2} (2x+5)^{\frac{1}{2}} \, dx$

$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left[(2x+5)^{\frac{3}{2}}\right]_{-\frac{5}{2}}^{2}$

$= \frac{1}{3}\left\{[2(2)+5]^{\frac{3}{2}} - \left[2\left(-\frac{5}{2}\right)+5\right]^{\frac{3}{2}}\right\}$

$= \frac{1}{3}\left[(9)^{\frac{3}{2}} - 0\right]$

$= 9$ units^2

∴ Area of shaded region = Area of ΔQST + Area of QPS

$= \frac{27}{8} + 9$

$= \boldsymbol{12\frac{3}{8}}$ **units²**

8. **Topic: Applications of Differentiation & Integration (Kinematics)**

(i)
$a_P = 1.5 \text{ m/s}^2$
$v_P = \int 1.5\,dt$
$= 1.5t + c_1$

"Particle P … moves with a constant acceleration of 1.5 m/s^2 …"

$v = \int a\,dt$

When $t = 0$, $v_P = 9$ m/s:

"Particle P starts with a velocity of 9 m/s …"

$9 = 1.5(0) + c_1$
$c_1 = 9$

$\therefore$ **Velocity of P, $v_P = 1.5t + 9$** – (1)

$a_Q = 1 + \frac{t}{2}$
$v_Q = \int \left(1 + \frac{t}{2}\right) dt$
$= t + \frac{1}{4}t^2 + c_2$

$v = \int a\,dt$

When $t = 0$, $v_Q = 0$:

"Particle Q starts from rest …"

$0 = 0 + \frac{1}{4}(0)^2 + c_2$
$c_2 = 0$

$\therefore$ **Velocity of Q, $v_Q = t + \frac{1}{4}t^2$** – (2)

(ii) From (i), $v_P = 1.5t + 9$
$s_P = \int (1.5t + 9)\,dt$
$= \frac{3}{4}t^2 + 9t + c_3$

$s = \int v\,dt$

When $t = 0$, $s_P = 0$:

"P and Q, leave a point O at the same time …"
"t seconds is the time since leaving O …"

$0 = \frac{3}{4}(0)^2 + 9(0) + c_3$
$c_3 = 0$

$\therefore$ **Distance traveled by P, $s_P = \frac{3}{4}t^2 + 9t$** –(3)

Note: s_P & s_Q are the displacements of P and Q from O but since $t \geq 0$ and there are no negative components in s_P & s_Q
$\Rightarrow s_P$ & s_Q are the distances travelled by P and Q.

From (i), $v_Q = t + \frac{1}{4}t^2$

$s = \int v\,dt$

$s_Q = \int \left(t + \frac{1}{4}t^2\right) dt$
$= \frac{1}{2}t^2 + \frac{1}{12}t^3 + c_4$

When $t = 0$, $s_Q = 0$:

"P and Q, leave a point O at the same time …"
"t seconds is the time since leaving O …"

$0 = \frac{1}{2}(0)^2 + \frac{1}{12}(0)^3 + c_4$
$c_4 = 0$

$\therefore$ **Distance traveled by Q, $s_Q = \frac{1}{2}t^2 + \frac{1}{12}t^3$**

(iii) When Q collides with P,

P and Q are at the same distance from O.

$$s_Q = s_P$$
$$\frac{1}{2}t^2 + \frac{1}{12}t^3 = \frac{3}{4}t^2 + 9t$$
$$12\left(\frac{1}{2}t^2 + \frac{1}{12}t^3\right) = 12\left(\frac{3}{4}t^2 + 9t\right)$$
$$6t^2 + t^3 = 9t^2 + 108t$$
$$t^3 - 3t^2 - 108t = 0$$
$$t(t^2 - 3t - 108) = 0$$
$$t(t - 12)(t + 9) = 0$$
$$t = 0 \text{ (reject)}, t = 12, t = -9 \text{ (reject)}.$$

Reject: $t = 0$ (P&Q at point O)
$t = -9$ (t is positive)

$\therefore$ Distance from O when Q collides with P $= \frac{3}{4}(12)^2 + 9(12)$
$=$ **216 m**

Sub $t = 12$ into (3)

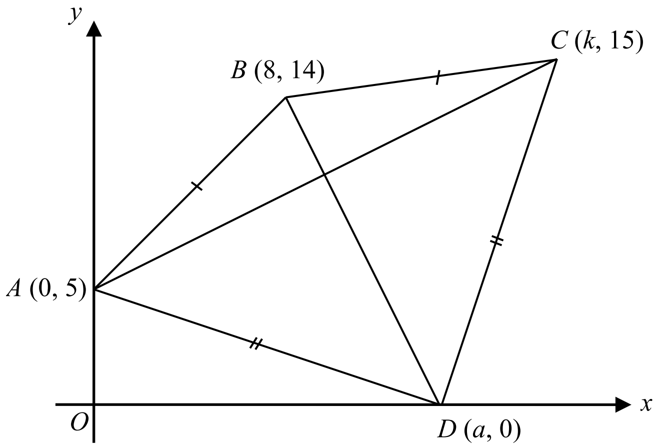
(iv) When $t = 12$,
$v_P = 1.5(12) + 9$
$=$ **27 m/s**

Sub $t = 12$ into (1)

$v_Q = 12 + \frac{1}{4}(12)^2$
$=$ **48 m/s**

Sub $t = 12$ into (2)

9. **Topic: Coordinate Geometry**



(i)

$$AB = BC$$
$$\sqrt{(8-0)^2+(14-5)^2} = \sqrt{(k-8)^2+(15-14)^2}$$
$$64 + 81 = (k-8)^2 + 1$$
$$(k-8)^2 = 144$$
$$k-8 = 12 \quad \text{or} \quad k-8 = -12$$
$$k = 20 \quad \text{or} \quad k = -4$$

Length of Line Segment $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Since k is positive, $k = 20$.

(ii) Let $D(a, 0)$.

D lies on x-axis ⇒ y-coordinate is 0

$$AD = CD$$
$$\sqrt{(a-0)^2+(0-5)^2} = \sqrt{(20-a)^2+(15-0)^2}$$
$$a^2 + 25 = 400 - 40a + a^2 + 225$$
$$40a = 600$$
$$a = 15$$

$\therefore$ **Coordinates of D = (15, 0)**

Equation of BD:

$$\frac{y-0}{x-15} = \frac{14-0}{8-15}$$
$$y - 0 = -2(x-15)$$
$$\mathbf{y = -2x + 30}$$

Equation of straight line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$:
$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} = \text{gradient } m$$

(iii) Area of ΔABC

$$= \frac{1}{2}\begin{vmatrix} 0 & 20 & 8 & 0 \\ 5 & 15 & 14 & 5 \end{vmatrix}$$
$$= \frac{1}{2}\left|[0 + (20)(14) + (8)(5)] - [(20)(5) + (8)(15) + 0]\right|$$
$$= 50 \text{ units}^2$$

Area of quadrilateral $ABCD$

$$= \text{Area of } \Delta ABC + \text{Area of } \Delta ADC$$
$$= 50 + \frac{1}{2}\begin{vmatrix} 0 & 15 & 20 & 0 \\ 5 & 0 & 15 & 5 \end{vmatrix}$$
$$= 50 + \frac{1}{2}\left|[0 + (15)(15) + (20)(5)] - [(15)(5) + 0 + 0]\right|$$
$$= 175 \text{ units}^2$$

$$\therefore \frac{\text{Area of } \Delta ABC}{\text{Area of quadrilateral } ABCD} = \frac{50}{175} = \frac{2}{7} \textbf{ (Shown)}$$

Area of triangle ABC ('Shoelace Method')

$$= \frac{1}{2}\begin{vmatrix} x_A & x_C & x_B & x_A \\ y_A & y_C & y_B & y_A \end{vmatrix}$$
$$= \frac{1}{2}(x_A y_c + x_C y_B + x_B y_A - x_C y_A - x_B y_C - x_A y_B)$$

Note: Coordinates must be taken in an <u>anticlockwise</u> direction.

10. **Topic: Partial Fractions, Differentiation& Integration**

(i)
$$\frac{3x^2+4x-20}{(2x+1)(x^2+4)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+4}$$
$$3x^2+4x-20 = A(x^2+4)+(Bx+C)(2x+1)$$

Sub $x = -\frac{1}{2}$,
$$3\left(-\tfrac{1}{2}\right)^2+4\left(-\tfrac{1}{2}\right)-20 = A\left[\left(-\tfrac{1}{2}\right)^2+4\right]+0$$
$$\boldsymbol{A = -5}$$

Sub $x = 0$,
$$-20 = 4A + C$$
$$-20 = 4(-5) + C$$
$$\boldsymbol{C = 0}\ \textbf{(Shown)}$$
$$3x^2+4x-20 = -5(x^2+4)+Bx(2x+1)$$
$$3x^2+4x-20 = (2B-5)x^2+Bx-20$$

Comparing coefficients of x,
$$\boldsymbol{B = 4}$$

(ii)
$$\frac{d}{dx}\ln(x^2+4) = \frac{1}{x^2+4}(2x) = \boldsymbol{\frac{2x}{x^2+4}}$$

$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$

(iii) From (i), we have
$$\frac{3x^2+4x-20}{(2x+1)(x^2+4)} = -\frac{5}{2x+1}+\frac{4x}{x^2+4}$$

Using result from (i)

$$\int \frac{3x^2+4x-20}{(2x+1)(x^2+4)}\,dx = \int\left(-\frac{5}{2x+1}+\frac{4x}{x^2+4}\right)dx$$
$$= -5\int\frac{1}{2x+1}dx + 2\int\left(\frac{2x}{x^2+4}\right)dx$$
$$= -5\left[\frac{1}{2}\ln(2x+1)\right]+2[\ln(x^2+4)]+c$$
$$= \boldsymbol{-\frac{5}{2}\ln(2x+1)+2\ln(x^2+4)+c,\ x>-\frac{1}{2}}$$

Using result from (ii)

11. **Topic: Trigonometric Functions,**
Further Trigonometric Identities (*R*-Formula)

(i) At A, $2 + 3\sin x$ is maximum:
$$\Rightarrow \sin x = 1$$
$$x = \frac{\pi}{2}$$
$$\Rightarrow y_{max} = 2+3(1) = 5$$

$\therefore$ **Coordinates of** $\boldsymbol{A = \left(\frac{\pi}{2},\ 5\right)}$

At B, $2 + 3\sin x$ is minimum:
$$\Rightarrow \sin x = -1$$
$$x = \frac{3\pi}{2}$$
$$\Rightarrow y_{min} = 2+3(-1) = -1$$

$\therefore$ **Coordinates of** $\boldsymbol{B = \left(\frac{3\pi}{2}, -1\right)}$

At C, $4\cos x$ is minimum:
$$\Rightarrow \cos x = -1$$
$$x = \pi$$
$$\Rightarrow y_{min} = 4(-1) = -4$$

$\therefore$ **Coordinates of** $\boldsymbol{C = (\pi, -4)}$

(ii) $4\cos x = 2 + 3\sin x$

$4\cos x - 3\sin x = 2 \quad -(1)$

Using R-formula,

$4\cos x - 3\sin x = R\cos(x+\alpha)$

$= R\cos\alpha\cos x - R\sin\alpha\sin x$

> R-Formula:
> $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$
> where
> $\tan\alpha = \frac{b}{a}$
> $R = \sqrt{a^2+b^2}$

Comparing coefficients:

$R\cos\alpha = 4 \quad -(2)$

$R\sin\alpha = 3 \quad -(3)$

$(2)^2 + (3)^2$:

$R^2(\cos^2\alpha + \sin^2\alpha) = 4^2 + 3^2$

> $\cos^2\theta + \sin^2\theta = 1$

$R^2 = 25$

$R = 5$ or -5(rejected)

$\frac{(3)}{(2)}$: $\tan\alpha = \frac{3}{4}$

$\alpha = \tan^{-1}\left(\frac{3}{4}\right)$

$= 0.6435$

≈ 0.644 (3 sig. fig.)

$\therefore \quad 4\cos x - 3\sin x = 5\cos(x + 0.644)$

From (1),

$4\cos x - 3\sin x = 2$

$\Rightarrow 5\cos(x+0.644) = 2$

$\cos(x + 0.644) = \frac{2}{5}$ where $\alpha = 0.644$, $k = \frac{2}{5}$.

(iii) $4\cos x = 2 + 3\sin x$

From (ii),

$\cos(x + 0.6435) = \frac{2}{5}$

> ***Hence question**: use result from (ii)

Basic $\angle\phi = 1.159$

$\therefore x + 0.6435 = 1.159,\ 2\pi - 1.159$

$x = 0.5155,\ 4.481$

$\approx 0.516,\ 4.48$ (3 sig. fig.)

$\therefore x$-coordinate of $D = 0.516$, x-coordinate of $E = 4.48$.