ADDITIONAL MATHEMATICS 4038/02
Paper 2 Suggested Solutions October/November 2009

1. **Topic: Further Trigonometric Identities**

(i)
$$\sin(A-B) = \frac{3}{8}$$
$$\sin A\cos B - \cos A\sin B = \frac{3}{8}$$
$$\frac{5}{8} - \cos A\sin B = \frac{3}{8}$$
$$\cos A\sin B = \frac{5}{8}-\frac{3}{8}$$
$$= \mathbf{\frac{1}{4}}$$

Addition Formula:
sin $(A–B)$ = sin A cos B – cos A sin B

(ii)
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$= \frac{5}{8}+\frac{1}{4}$$
$$= \mathbf{\frac{7}{8}}$$

Addition Formula:
sin $(A+B)$ = sin A cos B + cos A sin B

(iii)
$$\frac{\tan A}{\tan B} = \frac{\sin A}{\cos A} \div \frac{\sin B}{\cos B}$$
$$= \frac{\sin A}{\cos A}\times\frac{\cos B}{\sin B}$$
$$= \frac{\sin A\cos B}{\cos A\sin B}$$
$$= \frac{\frac{5}{8}}{\frac{1}{4}}$$
$$= \mathbf{2\frac{1}{2}}$$

Using ans from (ii)

Using ans from (i)

2. **Topic: Partial Fractions and Integration**

(i)
$$\frac{7}{2x^2-x-6} = \frac{7}{(2x+3)(x-2)}$$
$$\Rightarrow \frac{7}{(2x+3)(x-2)} = \frac{A}{2x+3}+\frac{B}{x-2}$$

Distinct linear factor:
$$\frac{\mathrm{P}(x)}{(ax+b)(cx-d)} = \frac{A}{ax+b}+\frac{B}{cx+d}$$

$$7 = A(x-2)+B(2x+3)$$
When $x = 2$, $7 = 0 + B(7)$
$$B = 1$$
When $x = -\frac{3}{2}$, $7 = A\left(-\frac{3}{2}-2\right)+0$
$$A = -2$$
$$\therefore \mathbf{\frac{7}{2x^2-x-6} = -\frac{2}{2x+3}+\frac{1}{x-2}}$$

***Hence question**:
From part (i)

(ii)
$$\int_3^9 \frac{7}{2x^2-x-6}\,\mathrm{d}x = \int_3^9\left(-\frac{2}{2x+3}+\frac{1}{x-2}\right)\mathrm{d}x$$
$$= \left[-\frac{2\ln(2x+3)}{2}+\ln(x-2)\right]_3^9$$
$$= [-\ln(21)+\ln 7]-[-\ln 9+\ln 1]$$
$$\approx -1.0986-(-2.1972)$$
$$= 1.0986$$
$$\approx \mathbf{1.10\ (3\ sig.\ fig.)}$$

3. **Topic: Indices, Logarithms and Factor Theorem**

(i)
$$u = 2^x$$
$$8^x - 2^{x+2} = 15$$
$$(2^3)^x - 2^x \times 2^2 = 15$$
$$(2^x)^3 - 4\times 2^x = 15$$
$$\therefore \mathbf{u^3 - 4u - 15 = 0}$$

(ii) Let $f(u) = u^3 - 4u - 15$

When $u = 3$, $f(3) = 3^3 - 4(3) - 15$
$= 0$

$\therefore$ By factor theorem, $(u - 3)$ is a factor.

$f(u) = (u-3)(u^2 + bu + 5)$

Compare coefficients of u: $-4 = 5 - 3b$
$3b = 9$
$b = 3$

$\therefore f(u) = (u-3)(u^2 + 3u + 5) = 0$

$\Rightarrow u - 3 = 0$ or $u^2 + 3u + 5 = 0$

$u = 3$ $\quad$ $u = \frac{-3\pm\sqrt{3^2-4(1)(5)}}{2}$

$= \frac{-3\pm\sqrt{-11}}{2}$ (rejected)

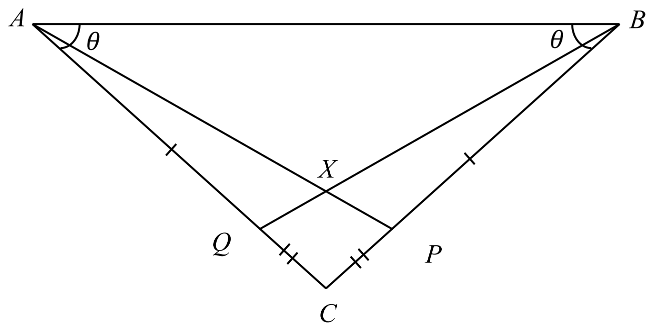
$\therefore$ u = 3 is the only real solution of this equation (Shown).

(iii)
$$u = 3$$
$$\Rightarrow 2^x = 3$$
$$\lg 2^x = \lg 3$$
$$x\lg 2 = \lg 3$$

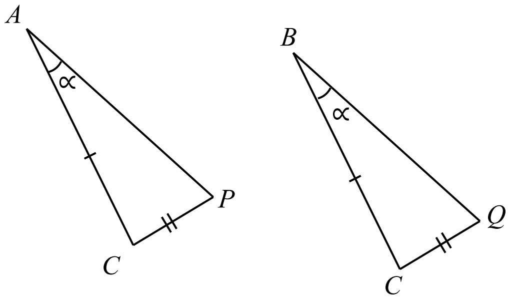
$\log_a (x)^n = n\log_a x$

$$x = \frac{\lg 3}{\lg 2}$$
$$\approx 1.584$$
$$\approx \mathbf{1.58 \text{ (3 sig. fig.)}}$$

4. **Topic: Plane Geometry**



(i) Given $\Delta\, ABC$ is an isosceles Δ,
Let $\angle CAB = \angle CBA = \theta$



In $\Delta\, ACP$ and $\Delta\, BCQ$,

$$AC = BC \quad \text{(Given)}$$
$$CP = CQ \quad \text{(Given)}$$
$$\angle ACP = \angle BCQ \quad \text{(Common)}$$

$\Rightarrow \Delta\, ACP$ is congruent to $\Delta\, BCQ$ (SAS).

Hence $\angle CAP = \angle CBQ = \propto$
$\Rightarrow \quad \angle XAB = \angle CAB - \angle CAP = \theta - \propto$
and $\angle XBA = \angle CBA - \angle CBQ = \theta - \propto$
$\therefore \angle XAB = \angle XBA$
Hence $\Delta\, XAB$ is isosceles Δ. **(Shown)**

(ii) Since $\Delta\, XAB$ is isosceles Δ, $XA = XB = a$ …………… (1)
Since $\Delta\, ACP$ is congruent to $\Delta\, BCQ$ (using result from part (i)),

$$AP = BQ$$
$$AX + XP = BX + XQ$$
$$a + XP = a + XQ \text{ (using result from part (i))}$$
$$XP = XQ$$
$$PX = QX \textbf{ (Shown)}$$

5. **Topic: Binomial Expansion**

(i)
$$\left(2 - \frac{x}{4}\right)^n = 2^n + {}^nC_1(2)^{n-1}\left(-\frac{x}{4}\right)^1 + {}^nC_2(2)^{n-2}\left(-\frac{x}{4}\right)^2 + \ldots$$
$$= 2^n + n(2)^{n-1}\left(-\frac{x}{2^2}\right) + \frac{n(n-1)}{2}(2)^{n-2}\left(\frac{x^2}{2^4}\right) + \ldots$$
$$= \mathbf{2^n - n2^{n-3}x + n(n-1)(2)^{n-7}x^2 + \ldots}$$

Expanding $(a+b)^n$:
$T_{r+1} = \binom{n}{r} a^{n-r} b^r$
Given in formula sheet:
$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$
$= \frac{n(n-1)\cdots(n-r+1)}{r!}$
$\therefore {}^nC_1 = \binom{n}{1} = n$
${}^nC_2 = \binom{n}{2} = \frac{n(n-1)}{2!}$

(ii)
$$(1+x)\left(2 - \frac{x}{4}\right)^n = a + bx^2$$
$$\Rightarrow (1+x)[\,2^n - n2^{n-3}x + n(n-1)(2)^{n-7}x^2 + \ldots] = a + bx^2$$
$$2^n - n2^{n-3}x + n(n-1)(2)^{n-7}x^2 + 2^n x - n2^{n-3}x^2 = a + bx^2$$

Compare coefficients of x^0: $\quad 2^n = a$ …………………… (1)

Compare coefficients of x^1: $\quad -n2^{n-3} + 2^n = 0$

$$2^{n-3}[-n + 2^3] = 0$$

$2^{n-3} = 0$ (reject) $\quad$ or $\quad -n + 8 = 0$

$$\mathbf{n = 8}$$

(iii) Compare coefficients of x^2: $n(n-1)(2)^{n-7} - n2^{n-3} = b$ ………….. (2)

Sub $n = 8$ into (1), $\quad a = 2^8$
$\quad = \mathbf{256}$

Sub $n = 8$ into (2), $\quad b = 8(7)2^1 - 8(2)^5$
$\quad = \mathbf{-144}$

6. **Topic: Trigonometric Functions and Area under curve**

(i) $y = 1 + 2\cos x$

When $y = 0$, $1 + 2\cos x = 0$

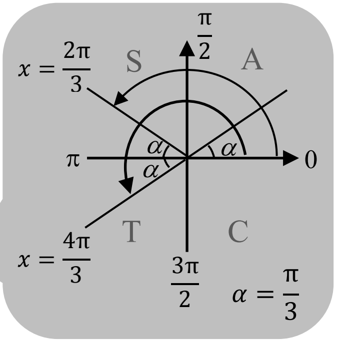
$\cos x = -\frac{1}{2}$

Basic $\angle \alpha = \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$

$\therefore x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}, \frac{4\pi}{3}$



$\therefore$ x-coordinate of A is $\frac{2\pi}{3}$ (Shown) and x-coordinate of B is $\frac{4\pi}{3}$.

(ii) Area of the shaded region

$= \int_0^{\frac{2\pi}{3}} (1 + 2\cos x)\,dx - \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2\cos x)\,dx$

Using values of A and B found in (i) as the limits

$= [x + 2\sin x]_0^{\frac{2\pi}{3}} - [x + 2\sin x]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$

$= \left[\frac{2\pi}{3} + 2\sin\frac{2\pi}{3} - 0\right] - \left[\frac{4\pi}{3} + 2\sin\frac{4\pi}{3} - \frac{2\pi}{3} - 2\sin\frac{2\pi}{3}\right]$

$= \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) - \frac{4\pi}{3} - 2\left(-\frac{\sqrt{3}}{2}\right) + \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right)$

$= \sqrt{3} + \sqrt{3} + \sqrt{3}$

≈ 5.196

$\approx$ 5.20 units^2 (3sig. fig.)

7. **Topic: Modulus Functions**

(i) $y = |3x - 5| - 2$

When $x = 0$, $y = |0 - 5| - 2$

$= 3$

When $y = 0$, $|3x - 5| - 2 = 0$

$3x - 5 = 2$ or $3x - 5 = -2$

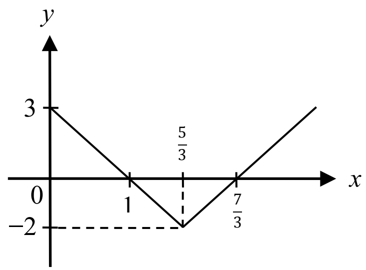
$3x = 7$ $\quad$ $3x = 3$

$x = \frac{7}{3}$ $\quad$ $x = 1$

$= 2\frac{1}{3}$

$\therefore$ Coordinates of all the points meeting the axes are (0, 3), $\left(2\frac{1}{3}, 0\right)$ and (1, 0).

(ii) $y = |3x - 5| - 2$



(iii) $x = |3x - 5| - 2$

$x + 2 = |3x - 5|$

Solving modular equations: $|f(x)| = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ -f(x) \text{ if } f(x) < 0 \end{cases}$

$x + 2 = 3x - 5$ or $-(x + 2) = 3x - 5$

$2x = 7$ $\quad$ $-x - 2 = 3x - 5$

$x = 3.5$ $\quad$ $4x = 3$

$x = 0.75$

8. **Topic: Applications of Differentiation (Kinematics)**

(i) $s = 400\left(1 - e^{-\frac{t}{10}}\right) - 16t$

$v = \frac{ds}{dt}$

$= 400\left(\frac{1}{10}e^{-\frac{t}{10}}\right) - 16$

$= \mathbf{40e^{-\frac{t}{10}} - 16}$

(ii) $a = \frac{dv}{dt}$

$= 40\left(-\frac{1}{10}\right)e^{-\frac{t}{10}}$

$= \mathbf{-4e^{-\frac{t}{10}}}$

“*t* seconds after passing *A*”

(iii) At *A*, $t = 0$, $\therefore v = 40e^0 - 16$

$= \mathbf{24\ m/s}$

“coming to a rest at a point *B*”

(iv) At *B*, $v = 0$, $40e^{-\frac{t}{10}} - 16 = 0$

$40e^{-\frac{t}{10}} = 16$

$e^{-\frac{t}{10}} = 0.4$

$\ln e^{-\frac{t}{10}} = \ln 0.4$

$-\frac{t}{10} = \ln 0.4$

$t = 9.1629$

≈ 9.163 seconds **(Shown)**

(v) Total distance $= 400\left(1 - e^{-\frac{9.163}{10}}\right) - 16(9.163)$

$= 93.393$ m

Sub $t = 9.163$ from part (iv) into s

Average speed of the motorcycle for the journey from *A* to *B*

$= \frac{\text{Total distance}}{\text{Total time taken}}$

$= \frac{93.393}{9.163}$

≈ 10.192

$\approx$ **10.2 m/s (3 sig. fig.)**

9. **Topic: Coordinate Geometry (Circles)**

Given $x^2 + y^2 + 2gx + 2fy + c = 0$ ………………………………........ (1)

(i) Equation of the circle: $(x - 2)^2 + (y + 1)^2 = 5^2$

$x^2 - 4x + 4 + y^2 + 2y + 1 - 25 = 0$

$x^2 - 4x + y^2 + 2y - 20 = 0$

$x^2 + y^2 - 4x + 2y - 20 = 0$ ………………… (2)

Comparing coefficients between (1) and (2)

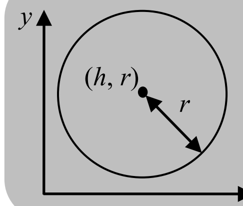
$\Rightarrow -4 = 2g$

$\mathbf{g = -2}$

$\Rightarrow 2 = 2f$

$\mathbf{f = 1}$

$\Rightarrow \mathbf{c = -20}$



Equation of circle (Standard Form):
$(x - h)^2 + (y - k)^2 = r^2$

(ii) Since AC is parallel to x-axis ⇒ y-coordinate of $A = -1$

Sub $y = -1$ into (2), $x^2 + (-1)^2 - 4x + 2(-1) - 20 = 0$

$$x^2 - 4x - 21 = 0$$
$$(x-7)(x+3) = 0$$

$x - 7 = 0$ or $x + 3 = 0$

$x = 7$ (rejected) $\quad x = -3$

∴ Coordinates of A are $(-3, -1)$

(iii) Gradient of AB = Gradient of OA

$$= \frac{-1-0}{-3-0}$$
$$= \frac{1}{3}$$

Equation of AB: $y + 1 = \frac{1}{3}(x+3)$

Equation of line with gradient m and point (x, y): $y - y_1 = m(x - x_1)$

$$\boldsymbol{y = \frac{1}{3}x} \quad \text{.......................... (3)}$$

(iv) Sub (2) into (1),

$$x^2 + \left(\frac{1}{3}x\right)^2 - 4x + 2\left(\frac{1}{3}x\right) - 20 = 0$$
$$x^2 + \frac{1}{9}x^2 - 4x + \frac{2}{3}x - 20 = 0$$
$$9x^2 + x^2 - 36x + 6x - 180 = 0$$
$$10x^2 - 30x - 180 = 0$$
$$x^2 - 3x - 18 = 0$$
$$(x-6)(x+3) = 0$$

$x - 6 = 0$ or $x + 3 = 0$

$x = 6 \quad x = -3$ (rejected)

Sub $x = 6$ into (2), $y = \frac{1}{3}(6)$

$= 2$

∴ Coordinates of B are (6, 2)

10. **Topics: Applications of Differentiation (Stationary Points)**

$$\frac{d^2y}{dx^2} = 6x - 6$$
$$\frac{dy}{dx} = \int (6x - 6)\ dx$$
$$= 3x^2 - 6x + c$$

When $x = 3$, $\frac{dy}{dx} = 12$

⇒ $12 = 3(9) - 6(3) + c$

$c = 3$

$$\therefore \frac{dy}{dx} = 3x^2 - 6x + 3$$
$$y = \int 3x^2 - 6x + 3\ dx$$
$$= x^3 - 3x^2 + 3x + d$$

Sub (3, 10)

⇒ $10 = 27 - 27 + 9 + d$

$d = 1$

$$\therefore y = x^3 - 3x^2 + 3x + 1 \quad \text{........................ (1)}$$

For stationary point on the curve: $\frac{dy}{dx} = 0$

$$3x^2 - 6x + 3 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

Sub $x = 1$ into (1), $y = 1 - 3 + 3 + 1$

$= 2$

∴ Coordinates of the stationary point are (1, 2)

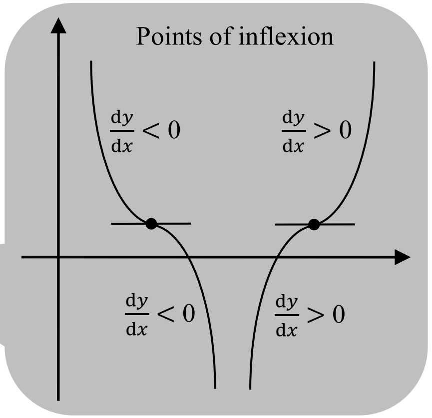
For (1, 2), using the 1st derivative test

x	0.9	1	1.1
$\frac{dy}{dx}$	+	O	+
	/	—	/

When $x = 0.9$, $\frac{dy}{dx} = 3(0.9)^2 - 6(0.9) + 3$
$= 0.03 > 0$

When $x = 1.1$, $\frac{dy}{dx} = 3(1.1)^2 - 6(1.1) + 3$
$= 0.03 > 0$

∴ (1, 2) is a point of inflexion.



11. **Topics: Further Trigonometric Identities (*R*-formula)**

(i) Given $\angle COD = \theta$

$$\begin{aligned}
\angle AOB &= \angle AOD - \angle COD \\
&= 90° - \theta \\
\angle OAB &= 90° - \angle AOB \\
&= 90° - (90° - \theta) \\
&= \theta \\
\angle BAO &= \theta \\
\cos\theta &= \frac{AB}{17} \\
AB &= 17\cos\theta \\
\sin\theta &= \frac{OB}{17} \\
OB &= 17\sin\theta \\
\cos\theta &= \frac{OC}{31} \\
OC &= 31\cos\theta \\
\therefore BC &= OC - OB \\
&= 31\cos\theta - 17\sin\theta \\
\sin\theta &= \frac{CD}{31} \\
CD &= 31\sin\theta \\
\therefore AB + BC + CD &= 17\cos\theta + 31\cos\theta - 17\sin\theta + 31\sin\theta \\
&= (48\cos\theta + 14\sin\theta) \text{ cm } \textbf{(Shown)}
\end{aligned}$$

(ii) Given $AB + BC + CD = 49$ (Proved in part (i))

$\Rightarrow 48\cos\theta + 14\sin\theta = 49$ …………………… (1)

Using R-formula,

$48\cos\theta + 14\sin\theta = R\cos(\theta - \alpha)$

$= R[\cos\theta\cos\alpha + \sin\theta\sin\alpha]$

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

R-Formula: $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$ where $\tan\alpha = \frac{b}{a}$, $R = \sqrt{a^2 + b^2}$

Comparing coefficients, $48 = R\cos\alpha$ ………………. (2)

$14 = R\sin\alpha$ ……………….. (3)

$\frac{(3)}{(2)}$: $\frac{R\sin\alpha}{R\cos\alpha} = \frac{14}{48}$

$\tan\alpha = 0.29166$

$\alpha = 16.26°$

$(2)^2 + (3)^2$: $R^2\cos^2\alpha + R^2\sin^2\alpha = 48^2 + 14^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 2500$

$R^2 = 2500$ ($\cos^2\theta + \sin^2\theta = 1$)

$R = 50$ or -50 (rejected)

$\therefore 48\cos\theta + 14\sin\theta = 50\cos(\theta - 16.26°)$ … (4)

Sub (4) into (1), $50\cos(\theta - 16.26°) = 49$

$\cos(\theta - 16.26°) = 0.98$

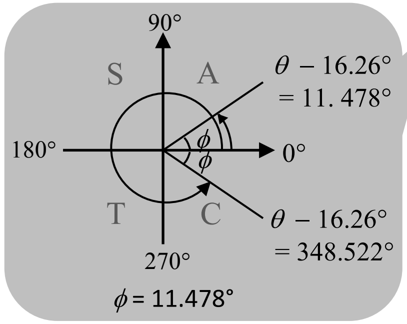
Basic $\angle\,\phi = 11.478°$

$\theta - 16.26° = 11.478°, 360° - 11.478°$

$\theta = 23.73°, 364.78°$

$= 27.73°, 4.78°$

$\approx$ **4.8°, 27.7° (1 d.p.)**



(iii) Maximum value of $AB + BC + CD = 50$

When $\cos(\theta - 16.26°) = 1$

$\Rightarrow \theta - 16.26° = 0°$

$\theta = 16.26°$

$\approx$ **16.3° (1 d.p.)**

Max/min values of $R\cos(\theta - \alpha) = \pm R$