ADDITIONAL MATHEMATICS 4038/01
Paper 1 Suggested Solutions October/November 2009

1. **Topic: Polynomials**

Let $f(x) = 2x^3 + ax^2 + bx + 3$

$f(1) = 0$

$\Rightarrow 2(1)^3 + a(1)^2 + b(1) + 3 = 0$

$2 + a + b + 3 = 0$

$a = -5 - b \quad -(1)$

$f(-2) = 15$

$\Rightarrow 2(-2)^3 + a(-2)^2 + b(-2) + 3 = 15$

$-16 + 4a - 2b + 3 = 15$

$4a - 2b = 28$

$2a - b = 14 \quad -(2)$

Factor Theorem:
$f(a) = 0 \Leftrightarrow (x - a)$ is a factor of $f(x)$

Remainder Theorem:
$f(x)$ divided by $(x - a)$
$\Rightarrow$ remainder is $f(a)$

Sub (1) into (2),

$2(-5 - b) - b = 14$

$-10 - 2b - b = 14$

$-3b = 24$

$b = -8$

Sub $b = -8$ into (1),

$a = -5 + 8$

$= 3$

$\therefore$ **$a = 3$ and $b = -8$**

2. **Topics: Applications of Differentiation (Increasing and Decreasing Functions)**

$y = \frac{\ln x}{x}$

$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}$

$= \frac{1 - \ln x}{x^2}$

Quotient rule:
For $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dv} - u\frac{dv}{dx}}{v^2}$

Since y is an increasing function of x, $\frac{dy}{dx} > 0$

$\Rightarrow 1 - \ln x > 0$ as $x^2 > 0$

$-\ln x > -1$

$\ln x < 1$

$x < e^1$

$\therefore$ **The set of values of $x = \{ x: 0 < x < e, x \in \mathbb{R}\}$**

3. **Topic: Surds**

$x\sqrt{24} = x\sqrt{3} + \sqrt{6}$

$x\sqrt{24} - x\sqrt{3} = \sqrt{6}$

$x(\sqrt{24} - \sqrt{3}) = \sqrt{6}$

$x = \frac{\sqrt{6}}{(\sqrt{24} - \sqrt{3})} \times \frac{(\sqrt{24} + \sqrt{3})}{(\sqrt{24} + \sqrt{3})}$

$= \frac{\sqrt{6}\sqrt{24} + \sqrt{6}\sqrt{3}}{24 - 3}$

$= \frac{\sqrt{144} + \sqrt{18}}{21}$

$= \frac{12 + \sqrt{9 \times 2}}{21}$

$= \frac{12 + 3\sqrt{2}}{21}$

$= \frac{4 + \sqrt{2}}{7}$

Rationalising the denominator:
$\frac{1}{\sqrt{a} + \sqrt{b}} \times \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{a - b}$

$\therefore$ **$a = 4, b = 2$**

4. **Topic: Logarithms**

(i)
$$\lg(x+14)-\lg(x-2) = 2\lg 5$$
$$\lg\left(\frac{x+14}{x-2}\right) = \lg 5^2$$
$$\frac{x+14}{x-2} = 25$$
$$x+14 = 25(x-2)$$
$$x+14 = 25x-50$$
$$24x = 64$$
$$\boldsymbol{x = 2\tfrac{2}{3}}$$

$n\log_a x = \log_a x^n$

$\log_a \frac{x}{y} = \log_a x - \log_a y$

(ii)
$$\log_2 y + \log_4 y = 6$$
$$\log_2 y + \frac{\log_2 y}{\log_2 4} = 6$$
$$\log_2 y + \frac{\log_2 y}{2} = 6$$
$$2\log_2 y + \log_2 y = 12$$
$$3\log_2 y = 12$$
$$\log_2 y = 4$$
$$y = 2^4$$
$$= \mathbf{16}$$

Changing of base: $\log_a N = \frac{\log_b N}{\log_b a}$

5. **Topics: Applications of Differentiation (Gradients, Tangents & Normal)**

$$y = 1 - 3\tan x$$
When curve intersects y-axis, $x = 0$,
$$y = 1 - 3\tan 0$$
$$= 1$$
$$\frac{dy}{dx} = -3\sec^2 x$$
When $x = 0$, $\frac{dy}{dx} = -3\sec^2 0$
$$= -3$$
$\therefore$ Gradient of tangent $= -3$

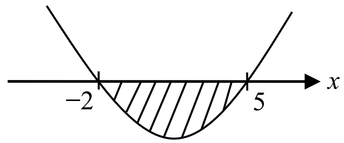
Gradient of normal $= \frac{1}{3}$

Equation of normal: $y - 1 = \frac{1}{3}(x-0)$
$$y = \frac{1}{3}x + 1 \qquad -(1)$$
Sub $(k, 3)$ into (1), $3 = \frac{1}{3}k + 1$
$$\frac{1}{3}k = 2$$
$$k = 6$$

6. **Topic: Quadratic Functions and Inequalities**

(i) $y = 2x^2 - 6x + c$

When $c = -20$, $2x^2 - 6x - 20 \le 0$
$$x^2 - 3x - 10 \le 0$$
$$(x-5)(x+2) \le 0$$



The set of values of x = {x: $-2 \le x \le 5$, $x \in \mathbb{R}$}

(ii)
$$y = 2x^2 - 6x + c \quad -(1)$$
$$y + 2x = 8 \quad -(2)$$

Sub (1) into (2),
$$2x^2 - 6x + c + 2x = 8$$
$$2x^2 - 4x + c - 8 = 0$$

$a = 2$, $b = -4$, $c = c - 8$

Since the line is a tangent to the curve, $b^2 - 4ac = 0$
$$(-4)^2 - 4(2)(c-8) = 0$$
$$16 - 8c + 64 = 0$$
$$8c = 80$$
$$\boldsymbol{c = 10}$$

7. **Topic: Coordinate Geometry**

$$x^2 + 2y^2 + 5x = 68 \quad -(1)$$
$$2y + 3x = 9$$
$$2y = 9 - 3x$$
$$y = \frac{9-3x}{2} \quad -(2)$$

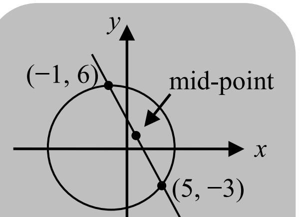
Sub (2) into (1),
$$x^2 + 2\left(\frac{9-3x}{2}\right)^2 + 5x = 68$$
$$x^2 + \frac{2(9-3x)^2}{2^2} + 5x = 68$$
$$x^2 + \frac{81-54x+9x^2}{2} + 5x = 68$$
$$2x^2 + 81 - 54x + 9x^2 + 10x = 136$$
$$11x^2 - 44x - 55 = 0$$
$$x^2 - 4x - 5 = 0$$
$$(x+1)(x-5) = 0$$
$$x = -1 \quad \text{or} \quad x = 5$$

Sub $x = -1$ into (2), $y = \frac{9-3(-1)}{2} = 6$

Sub $x = 5$ into (2), $y = \frac{9-3(5)}{2} = -3$

The coordinates of intersection points
⇒ $(-1, 6)$ and $(5, -3)$

$\therefore$ Mid-point $= \left(\frac{-1+5}{2}, \frac{6-3}{2}\right)$
$= \mathbf{(2, 1.5)}$



Mid-point of (x_1, y_1) and (x_2, y_2): $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

8. **Topic: Further Trigonometric Identities (Factor and Double Angle formula)**

(i) $\cos 3x - \cos x = -4\sin^2 x \cos x$

L.H.S. $= -2\sin\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)$
$= -2\sin 2x \sin x$
$= -2(2\sin x\cos x)\sin x$
$= -4\sin^2 x\cos x$
= R.H.S. **(Shown)**

Factor Formula:
$\cos A + \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

Double Angle Formula:
$\sin 2A = 2\sin A\cos A$

(ii)
$$\cos 3x + 2\cos x = 0$$
$$\cos 3x - \cos x = -3\cos x$$
$$-4\sin^2 x\cos x = -3\cos x$$
$$4\sin^2 x\cos x - 3\cos x = 0$$
$$\cos x(4\sin^2 x - 3) = 0$$

Using proof in (i)

$\cos x = 0$ or $4\sin^2 x - 3 = 0$

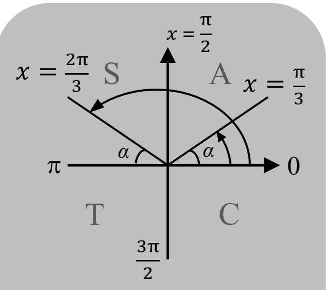
$x = \frac{\pi}{2}$ $\quad$ $\sin^2 x = \frac{3}{4}$

$\sin x = \pm\sqrt{\frac{3}{4}}$

Basic angle $\alpha = \frac{\pi}{3}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}$

$$\boldsymbol{\therefore x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}}$$



$\alpha = \frac{\pi}{3}$
N.B.: Basic angle is +ve and acute.

9. **Topic: Trigonometric Functions**

$f(x) = 3\sin\left(\frac{x}{3}\right) - 1$

(i) $-1 \leq \sin\left(\frac{x}{3}\right) \leq 1$

> The values of *a* sin *bx* lie between *a* and –*a*.

$-3 \leq 3\sin\left(\frac{x}{3}\right) \leq 3$

$-3 - 1 \leq 3\sin\left(\frac{x}{3}\right) - 1 \leq 3 - 1$

$-4 \leq 3\sin\left(\frac{x}{3}\right) - 1 \leq 2$

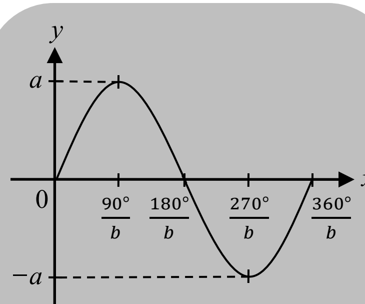
Maximum value of $f(x)$ = **2**

Minimum value of $f(x)$ = **−4**

(ii) Amplitude of f = **3**

(iii) Period of f $= \frac{360°}{\frac{1}{3}}$

$=$ **1080°**



Amplitude = *a*

Period = $\frac{360°}{b}$

(iv) $3\sin\left(\frac{x}{3}\right) - 1 = 0$

$\sin\left(\frac{x}{3}\right) = \frac{1}{3}$

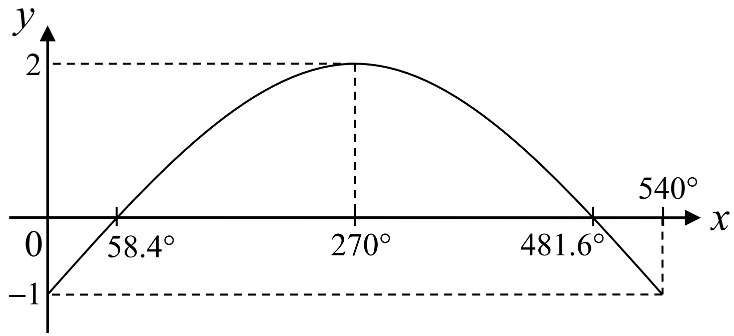
Basic ∠ = 19.47°

$\frac{x}{3} = 19.47°$

$x = 58.41°$

$\approx$ **58.4° (1 d.p.)**

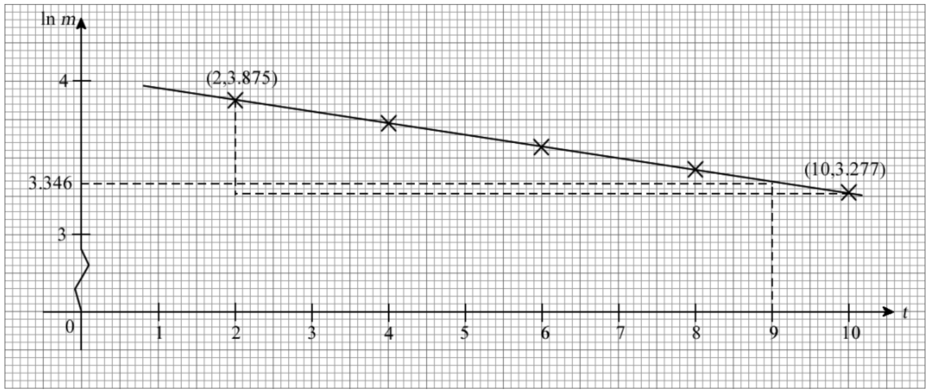
(v) $y = 3\sin\left(\frac{x}{3}\right) - 1$



10. **Topic: Straight Line Graphs/ Linear Law**

(i)

t	2	4	6	8	10
ln *m*	3.875	3.725	3.575	3.424	3.277



(ii)
$$m = m_o e^{-kt}$$
$$\ln m = \ln m_o e^{-kt}$$
$$= \ln m_o + \ln e^{-kt}$$
$$= \ln m_o - kt \ln e$$
$$\ln m = \ln m_o - kt$$
$$\Rightarrow \text{Gradient} = -k$$
$$\frac{3.277-3.875}{10-2} = -k$$
$$k = 0.07475$$
$$\approx 0.0748 \text{ (3 s.f.)}$$
$$\Rightarrow \ln m_o = 4.04 \text{ (intercept on } \ln m \text{ axis)}$$
$$m_o = e^{4.04}$$
$$= 56.82$$
$$\approx \mathbf{56.8 \text{ (3 s.f.)}}$$

(iii) When $m = \frac{1}{2} m_o$, $\ln m = \ln\left(\frac{56.82}{2}\right)$
$$\approx 3.346$$

From graph, t = **9 hours**

11. **Topic: Coordinate Geometry**

(i)
$$\text{Gradient of } AD = \frac{6+4}{0-2}$$
$$= -4$$
$$\text{Gradient of } AB = \frac{1}{4}$$

Equation of line with gradient m and point (x_1, y_1): $(y - y_1) = m(x - x_1)$

Equation of AB: $y - 6 = \frac{1}{4}(x - 0)$
$$\mathbf{y = \frac{1}{4}x + 6} \quad -(1)$$

(ii)
$$y = x \quad -(2)$$
Sub (1) into (2),
$$x = \frac{1}{4}x + 6$$
$$4x = x + 24$$
$$3x = 24$$
$$x = 8$$
$$\therefore y = 8$$
Coordinates of B are (8, 8).

Length of line segment $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

(iii) Let C be (x, y),
$$\text{Length of } DC = 2 \times \text{Length of } AB$$
$$\sqrt{(2-x)^2 + (-2-y)^2} = 2\sqrt{(0-8)^2 + (6-8)^2}$$
$$(2-x)^2 + (-2-y)^2 = 4(68)$$
$$(2-x)^2 + (-2-y)^2 = 272 \quad -(3)$$

Equation of CD: $y + 2 = \frac{1}{4}(x - 2)$
$$y = \frac{1}{4}x - \frac{5}{2} \quad -(4)$$

Sub (4) into (3),

$$4 - 4x + x^2 + 4 + 4\left(\frac{1}{4}x - \frac{5}{2}\right) + \left(\frac{1}{4}x - \frac{5}{2}\right)^2 = 272$$

$$4 - 4x + x^2 + 4 + x - 10 + \frac{1}{16}x^2 - \frac{5}{4}x + \frac{25}{16} - 272 = 0$$

$$\frac{17}{16}x^2 - \frac{17}{4}x - \frac{1071}{4} = 0$$

$$17x^2 - 68x - 4284 = 0$$

$$x^2 - 4x - 252 = 0$$

$$(x - 18)(x + 14) = 0$$

$x - 18 = 0$ or $x + 14 = 0$

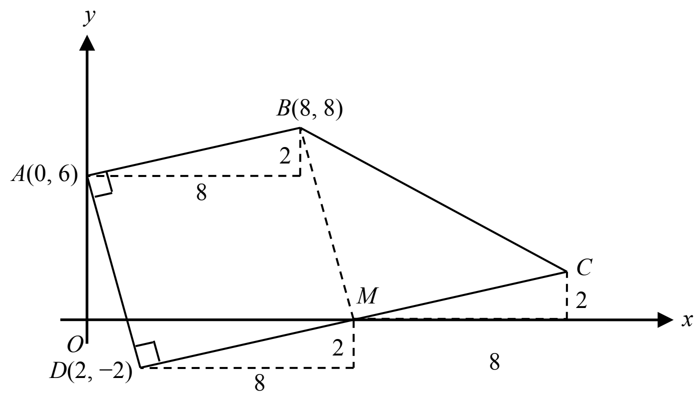
$x = 18$ $\quad$ $x = -14$ (rej.)

Sub $x = 18$ into (4), $y = \frac{1}{4}(18) - \frac{5}{2}$

$= 2$

∴ Coordinates of *C* are (18, 2)

<u>Alternate Method</u>



Given DC is twice the length of AB.

Let M be the mid-point of DC.

Then $DM = AB$ and DM // AB

Let M be (a, b) and $C(x, y)$

Then $b = -2 + 2 = 0$

$a = 2 + 8 = 10$

∴ M (10, 10)

Since M is the mid-point of DC,

$\frac{2+x}{2} = 10$ and $\frac{y+(-2)}{2} = 0$

∴ Coordinates of *C* are (18, 2)

Mid-point of (x_1, y_1) and (x_2, y_2):
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

(iv) Area of trapezium *ABCD* $= \frac{1}{2}\begin{vmatrix} 0 & 2 & 18 & 8 & 0 \\ 6 & -2 & 2 & 8 & 6 \end{vmatrix}$

$= \frac{1}{2}|0 + 4 + 144 + 48 - 12 + 36 - 16 - 0|$

$= \frac{1}{2}|204|$

$=$ **102 units²**

Area of quadrilateral *ABCD* ('Shoelace Method')

$= \frac{1}{2}\begin{vmatrix} x_A & x_D & x_C & x_B & x_A \\ y_A & y_D & y_C & y_B & y_A \end{vmatrix}$

$= \frac{1}{2}(x_A y_D + x_D y_C + x_C y_B + x_B y_A - x_D y_A - x_C y_D - x_B y_C - x_A y_B)$

Note: Coordinates must be taken in an anticlockwise direction.

12. **Topic: Differentiation and Integration**

(i) $y = (2x - 1)\sqrt{4x+1}$

$\frac{dy}{dx} = (2x-1)\frac{1}{2}(4x+1)^{-\frac{1}{2}}(4) + \sqrt{4x+1}\,(2)$

$= \frac{2(2x-1)}{\sqrt{4x+1}} + \frac{2\sqrt{4x+1}}{1}$

$= \frac{2(2x-1)+2(4x+1)}{\sqrt{4x+1}}$

$= \frac{4x-2+8x+2}{\sqrt{4x+1}}$

$= \boldsymbol{\frac{12x}{\sqrt{4x+1}}}$

Product Rule:
For $y = uv$,
$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

(ii) $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$2 = \frac{12(2)}{\sqrt{8+1}} \times \frac{dx}{dt}$

$2 = \frac{24}{3} \times \frac{dx}{dt}$

$\frac{dx}{dt} = \boldsymbol{\frac{1}{4}}$ **units per second**

Chain Rule:
$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

(iii) $\int_0^2 \frac{3x}{\sqrt{4x+1}}\,dx = \frac{1}{4}\int_0^2 \frac{12x}{\sqrt{4x+1}}\,dx$

$= \frac{1}{4}\left[(2x-1)\sqrt{4x+1}\right]_0^2$

$= \frac{1}{4}[3\sqrt{9} - (-1)(1)]$

$= \frac{1}{4}[9+1]$

$=$ **2.5**

***Hence question:**
Using answer from part (i)