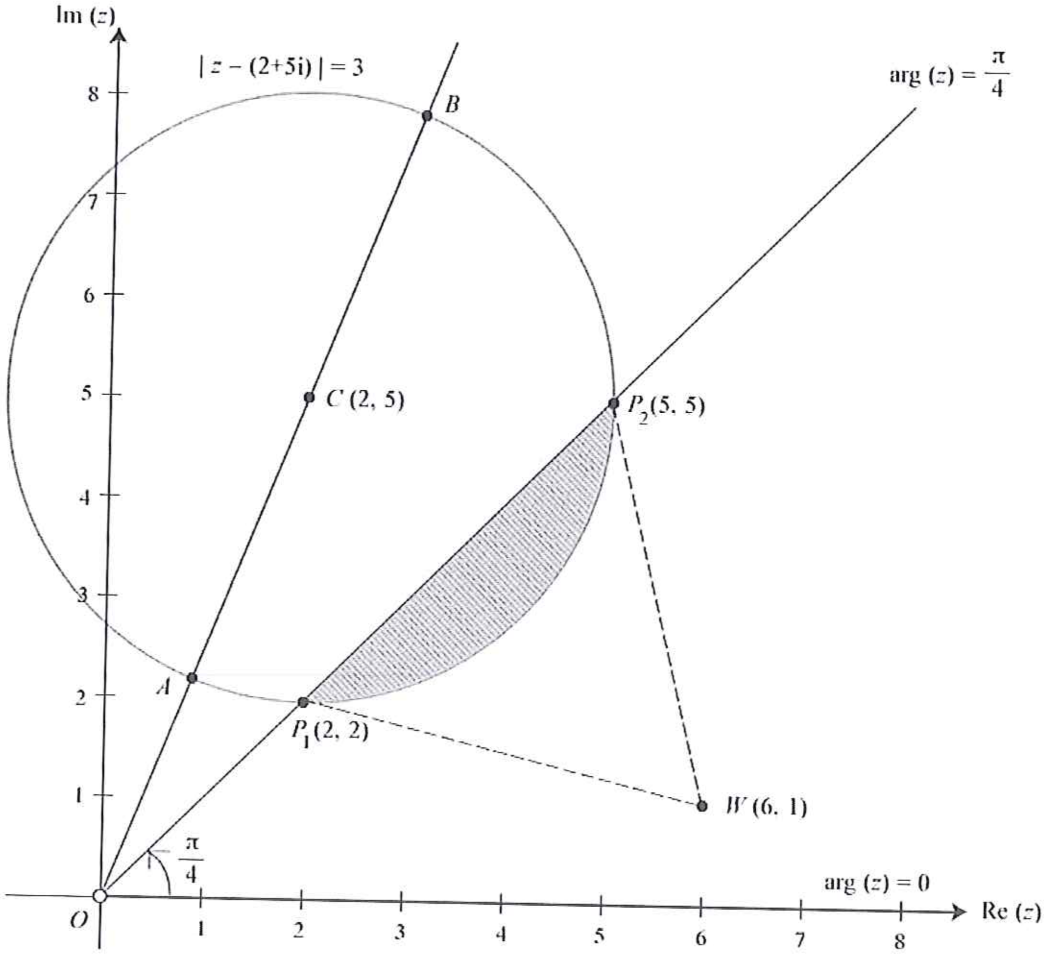
2011 H2 Maths P2: v1.2

1 (i)



2011 H2 Maths P2 :

1(i) Given $|z-2-5i| \leq 3.$

$|z-(2+5i)| \leq 3.$

(ii) Length of $OC = \sqrt{2^2+5^2}$

$= \sqrt{29}.$

Min value of $|z| = OA$

$= OC - AC$ {$\because AC$ = radius of circle $= 3$}.

$= \sqrt{29} - 3.$

Max value of $|z| = OB$

$= OC + CB$

$= \sqrt{29} + 3.$ {CB = radius of circle $= 3$}.

(iii) Let $W(6,1)$, $P_1(2,2)$, $P_2(5,5)$.

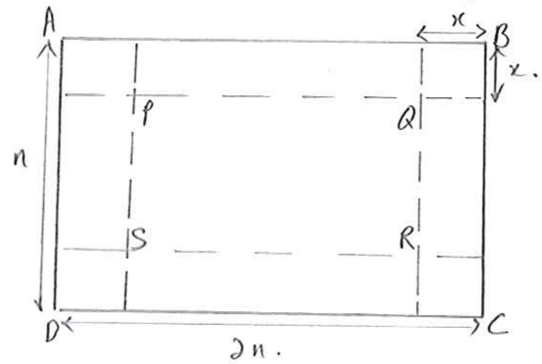
$WP_1 = \sqrt{(6-2)^2+(1-2)^2} = \sqrt{17}.$

$WP_2 = \sqrt{(6-5)^2+(1-5)^2} = \sqrt{17}.$

Max value of $|z-6-i|$ = max value of $|z-(6+i)|$

$= WP_1 = WP_2 = \sqrt{17}$ #.

2011 H2 Maths P2 :



Given n is a positive constant.

2. (i). Length of $SR = 2n - 2x$.

Length of $QR = n - 2x$.

$\therefore$ Volume of the open rectangular box:

$$V = (SR) \times (QR) \times x$$
$$= (2n-2x)(n-2x)x$$
$$= (2n^2 - 4nx - 2nx + 4x^2)x$$
$$= 2n^2x - 6nx^2 + 4x^3. \text{ (shown).}$$

$$\frac{dV}{dx} = 2n^2 - 12nx + 12x^2.$$

(ii) To find stationary value, $\frac{dV}{dx} = 0$.

$$2n^2 - 12nx + 12x^2 = 0.$$
$$6x^2 - 6nx + n^2 = 0.$$

$$\therefore x = \frac{6n - \sqrt{12n^2}}{12} \quad \text{or} \quad x = \frac{6n + \sqrt{12n^2}}{12}.$$

$$= \frac{6n - \sqrt{12}\,n}{12} \quad \text{or} \quad = \frac{6n + \sqrt{12}\,n}{12}.$$

$$= \frac{n}{12}(6 - \sqrt{12}) \qquad = \frac{n}{12}(6 + \sqrt{12})$$

$$= \frac{n}{6}(3 - \sqrt{3})_{*} \qquad = \frac{n}{6}(3 + \sqrt{3})_{*}$$

Note $2x < n$. (as seen from side AD)

$\Rightarrow x < \frac{n}{2}$

$\Rightarrow x = \frac{n}{6}(3 - \sqrt{3}) < \frac{n}{2}$. and $x = \frac{n}{6}(3 + \sqrt{3}) > \frac{n}{2}$.

$\therefore x = \frac{n}{6}(3 - \sqrt{3})$ is the only answer #.

2011 H2 Maths P2:

3 Given $f(x) = \ln(2x+1) + 3$, $x \in \mathbb{R}$, $x > -\frac{1}{2}$.

(i) Let $y = f(x)$ then $f^{-1}(y) = x$.

$$y = \ln(2x+1) + 3$$
$$y - 3 = \ln(2x+1)$$
$$2x+1 = e^{y-3}.$$
$$x = \frac{1}{2}(e^{y-3} - 1)$$
$$f^{-1}(y) = \frac{1}{2}(e^{y-3} - 1)$$
$$\therefore f^{-1}(x) = \frac{1}{2}(e^{x-3} - 1).$$

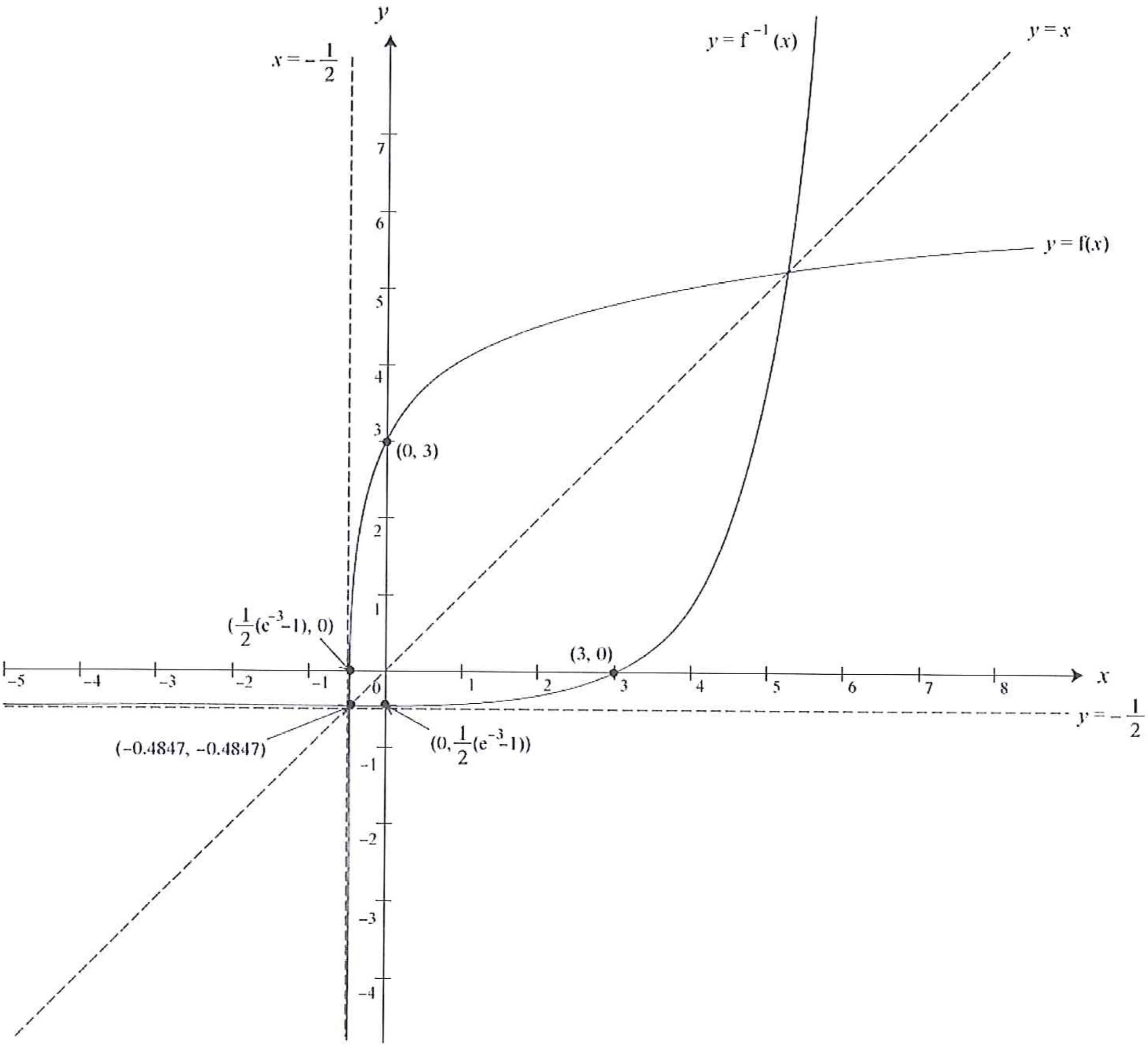
domain of $f^{-1}(x) = \mathbb{R}$
Range of $f^{-1}(x) = (-\frac{1}{2}, \infty)$

(ii). For curve $f(x)$:
Vertical asymptotes : $x = -\frac{1}{2}$.
y-intercept : when $x = 0$, $y = \ln(0+1) + 3 = 3$. $(0, 3)$.
x-intercept : when $y = 0$, $x = \frac{1}{2}(e^{-3} - 1)$. $(\frac{1}{2}(e^{-3}-1), 0)$

For curve. $f^{-1}(x)$:
horizontal asymptotes : $y = -\frac{1}{2}$.
y-intercept : when $x = 0$, $y = \frac{1}{2}(e^{-3} - 1)$ $(0, \frac{1}{2}(e^{-3}-1))$
x-intercept : when $y = 0$, $x = 3$ $(3, 0)$.

[See Next page for the diagram]

(iii). For $\ln(2x+1) = x - 3$.
$$\ln(2x+1) + 3 = x.$$
$$f(x) = x.$$
$$y = x.$$
$f(x)$ and $f^{-1}(x)$ intersect at $y = x$ i.e. $f(x) = f^{-1}(x) = x$.
Solving $f^{-1}(x) = x$.
$$\frac{1}{2}(e^{x-3} - 1) = x.$$
From G.C, we have
$$x = -0.48466 \quad \text{or} \quad x = 5.48188$$
$x = -0.4847, 5.482$ (corr to 4 sig fig).



y
x
$x = -\frac{1}{2}$
$y = f^{-1}(x)$
$y = x$
$y = f(x)$
$y = -\frac{1}{2}$
(0, 3)
(3, 0)
$(\frac{1}{2}(e^{-3}-1), 0)$
$(0, \frac{1}{2}(e^{-3}-1))$
(−0.4847, −0.4847)

2011 H2 Maths P2 :

4 (a) (i). $\int_0^n x^2 e^{-2x}\,dx.$, where $n > 0$.

$$= \int_0^n x^2 \, d(-\tfrac{1}{2}e^{-2x})$$

$$= \left[x^2(-\tfrac{1}{2}e^{-2x})\right]_0^n - \int_0^n (2x)(-\tfrac{1}{2}e^{-2x})\,dx.$$

$$= -\tfrac{1}{2}\left[x^2 e^{-2x}\right]_0^n + \int_0^n x e^{-2x}\,dx.$$

$$= -\tfrac{1}{2}\left[n^2 e^{-2n} - 0\right] + \int_0^n x \, d(-\tfrac{1}{2}e^{-2x})$$

$$= -\tfrac{1}{2}n^2 e^{-2n} + \left[x(-\tfrac{1}{2}e^{-2x})\right]_0^n - \int_0^n (1)(-\tfrac{1}{2}e^{-2x})\,dx.$$

$$= -\tfrac{1}{2}n^2 e^{-2n} - \tfrac{1}{2}\left[ne^{-2n} - 0\right] + \tfrac{1}{2}\int_0^n e^{-2x}\,dx.$$

$$= -\tfrac{1}{2}n^2 e^{-2n} - \tfrac{1}{2}ne^{-2n} - \tfrac{1}{4}\left[e^{-2x}\right]_0^n$$

$$= -\tfrac{1}{2}n^2 e^{-2n} - \tfrac{1}{2}ne^{-2n} - \tfrac{1}{4}e^{-2n} + \tfrac{1}{4} \ \#.$$

(ii) $$\int_0^\infty x^2 e^{-2x}\,dx = \lim_{n\to\infty}\left[\int_0^n x^2 e^{-2x}\,dx\right]$$

$$= \lim_{n\to\infty}\left[-\tfrac{1}{2}n^2 e^{-2n} - \tfrac{1}{2}ne^{-2n} - \tfrac{1}{4}e^{-2n} + \tfrac{1}{4}\right].$$

$$= \lim_{n\to\infty}\left[-\tfrac{1}{2}\left(\frac{n^2}{e^{2n}}\right) - \tfrac{1}{2}\left(\frac{n}{e^{2n}}\right) - \tfrac{1}{4}\left(\frac{1}{e^{2n}}\right) + \tfrac{1}{4}\right].$$

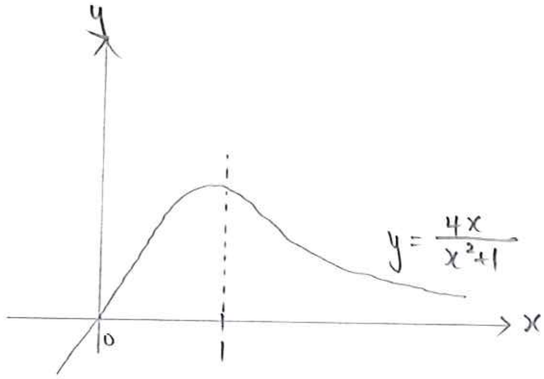
Note: e^{2n} increases exponentially faster than n^2, n and any constant.

$\Rightarrow \lim_{n\to\infty}\frac{n^2}{e^{2n}} = 0$, $\lim_{n\to\infty}\frac{n}{e^{2n}} = 0$ and $\lim_{n\to\infty}\frac{1}{e^{2n}} = 0$.

$$\therefore \int_0^\infty x^2 e^{-2x}\,dx = 0 + 0 + 0 + \tfrac{1}{4} = \tfrac{1}{4} \ \#.$$

2011 H2 Maths P2 :

4b Given $y = \frac{4x}{x^2+1}$.



Volume of the solid

$$V = \pi \int_0^1 y^2 \, dx.$$
$$= \pi \int_0^1 \left(\frac{4x}{x^2+1}\right)^2 dx.$$
$$= \pi \int_0^1 \frac{16x^2}{(x^2+1)^2} \, dx \quad \text{——— (1)}$$

Given $x = \tan\theta$.

when $x = 0$, $\theta = 0$

when $x = 1$, $\theta = \frac{\pi}{4}$.

$$\frac{dx}{d\theta} = \sec^2\theta$$
$$dx = \sec^2\theta \, d\theta.$$

From (1), we have

$$V = \pi \int_0^{\pi/4} \frac{16\tan^2\theta}{(\tan^2\theta+1)^2} \sec^2\theta \, d\theta$$
$$= \pi \int_0^{\pi/4} \frac{16\tan^2\theta}{(\sec^2\theta)^2} \sec^2\theta \, d\theta.$$
$$= \pi \int_0^{\pi/4} \frac{16\tan^2\theta}{\sec^2\theta} \, d\theta.$$
$$= 16\pi \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \, d\theta.$$
$$= 16\pi \int_0^{\pi/4} \sin^2\theta \, d\theta. \quad \text{(shown).}$$

Note that $\cos 2\theta = 1 - 2\sin^2\theta$.
$\Rightarrow \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$.

$$= 16\pi \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta) \, d\theta.$$
$$= 8\pi \left[\theta - \frac{1}{2}\sin 2\theta \right]_0^{\pi/4}.$$
$$= 8\pi \left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} - 0 \right].$$
$$= 8\pi \left[\frac{\pi}{4} - \frac{1}{2} \right].$$
$$= (2\pi^2 - 4\pi) \text{ unit}^3 \ \#.$$

2011 H2 Maths P2:

5 Given $X \sim N(\mu, \sigma^2)$

Given $P(X < 40.0) = 0.05$.

then $P\left(\frac{X-\mu}{\sigma} < \frac{40-\mu}{\sigma}\right) = 0.05$.

$P\left(Z < \frac{40-\mu}{\sigma}\right) = 0.05$.

$\frac{40-\mu}{\sigma} = \text{invNorm}(0.05)$.

$\frac{40-\mu}{\sigma} = -1.6449$

$40-\mu = -1.6449\sigma$

$\mu = 40 + 1.6449\sigma$ —— (1).

Given $P(X < 70.0) = 0.975$.

$P\left(\frac{X-\mu}{\sigma} < \frac{70-\mu}{\sigma}\right) = 0.975$.

$P\left(Z < \frac{70-\mu}{\sigma}\right) = 0.975$.

$\frac{70-\mu}{\sigma} = \text{InvNorm}(0.975)$.

$\frac{70-\mu}{\sigma} = 1.9599$.

$70-\mu = 1.9599\sigma$ —— (2).

sub (1) into (2).

$70-(40+1.6449\sigma) = 1.9599\sigma$.

$30 - 1.6449\sigma = 1.9599\sigma$

$\sigma = \frac{30}{3.6048}$

$= 8.3222$

≈ 8.32 # (corr to 3 sig fig)

sub $\sigma = 8.3222$ into (1).

$\mu = 40 + 1.6449(8.3222)$

$= 53.689$.

≈ 53.7 # (corr to 3 sig fig).

$\therefore \mu = 53.7$ and $\sigma = 8.32$ #

2011 H2 Maths P2:

6(i) A quota sample might be carried out by first determining a number of age groups within the desired range of ages, and then instructing each interviewer to interview a certain number of residents within a each age group.

(ii) A disadvantage is the possibility of bias in the selection process, as interviewers may tend to pick their interviewees who look cooperative.

(iii) Stratified sampling. However this is not realistic as it is difficult to pick a completely random sample of residents from each strata and get them to be interviewed as and when specifically desired.

2011 H2 Maths P2.

7(i) ① The probability, p, of a successful outcome is the same for each trial

② Each call is either successful or not successful.

7(ii) As the probability value of 0.7 given in this case is based on an average value rather than a fixed probability of success, this assumption may not hold.

7iii For $n=8$.

$R \sim Bin(8, 0.7)$.

$P(R \geq 6) = 1 - P(R \leq 5)$
$= 1 - \text{binomcdf}(8, 0.7, 5)$
$= 0.55177$
≈ 0.552 ((orr to 3 sig fig).

(iv) For $n = 40$.

$np = 40(0.7) = 28$
$nq = 40(0.3) = 12$

Since $np > 5$ and $nq > 5$.

$R \sim Bin(40, 0.7)$ can be approximated to be $R \sim N(28, 8.4)$.

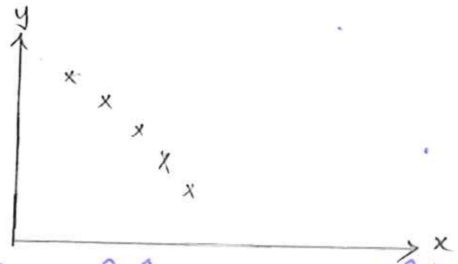
Note $E(R) = 40(0.7) = 28$.

$Var(R) = 40(0.7)(0.3) = 8.4$.

$P(R < 25) \xrightarrow[\text{Correction}]{\text{Continuity}} P(R < 24.5)$
$= \text{normalcdf}(-1E99, 24.5, 28, \sqrt{8.4})$
$= 0.11359$
≈ 0.114 ((orr to 3 sig fig).

2011 H2 Maths P2.

8



(ii) From GC, $r = -0.99231$
≈ -0.992 (corr to 3 sig fig).

From the plot in G.C, the data appears to suggest a non-linear relationship between x and y. Hence $y = c + dx$ may not be the best model even though the value of r obtained is high.

(iii). By computing the r-value of x^2 & y.
we have $r = -0.99998$
≈ -1.00 (corr to 3 sig fig)

The almost perfect r-value of -1 indicates that $y = a + bx^2$ is a better model compared to $y = c + dx$ with a relatively lower r-value.

(iv)

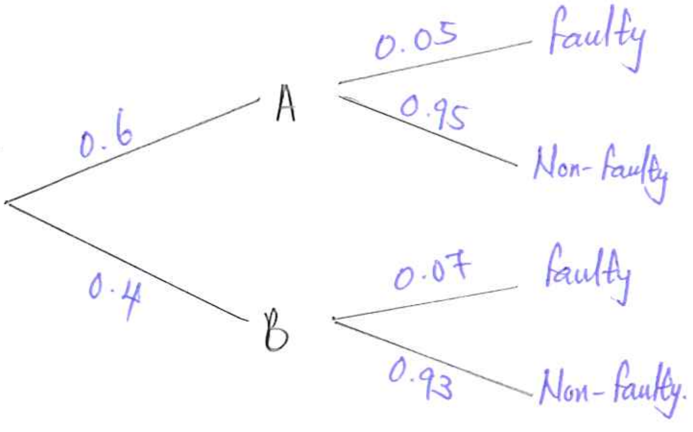
x^2	4	6.25	9	12.25	16	20.25	25
y	18.8	16.9	14.5	11.7	8.6	4.9	0.8

From GC: $y = 22.23 - 0.8562x^2$.

When $x = 3.2$, $y = 22.23 - 0.8562(3.2)^2$
$= 13.462$
≈ 13.5 (corr to 3 sig fig).

2011 H2 Maths P2.

9



(i) (a) $P(\text{a lens is faulty}) = 0.6(0.05) + 0.4(0.07)$
$= 0.058$ #

(b) $P(\text{it was made by A} \mid \text{it is faulty})$
$= \dfrac{P(\text{It was made by A and it is faultey})}{P(\text{it was faulty})}$

$= \dfrac{0.6(0.05)}{0.058}$
$= 0.5172$
≈ 0.517 ((cor to 3 sig fig).

(ii) (a) $P(\text{exactly one of them is faulty})$
$= P(L_1 \text{ faulty}) + P(L_2 \text{ faulty})$
$= 0.058(1-0.058) + (1-0.058)\,0.058.$
$= 0.109272$
≈ 0.109 ((cor to 3sig fig).

(b)

Let X : both lenses made by A.
Y : exactly one lens is faulty.
$P(X/Y) = P(X \cap Y) / P(Y)$
$= \dfrac{(0.6)(0.05)(0.6)(0.95)\ 2!}{0.109}$

$= 0.31376$
≈ 0.314 ((cor to 3 sig fig).

2011 H2 Maths P2.

10(i). Given $T \sim N(38.0, 5^2)$.

Given sample of size n, $\bar{T} \sim N(38.0, \frac{5^2}{n})$.

Null hypothesis H_0 : $\mu = 38.0$
Alternative hypothesis H_1 : $\mu < 38.0$

Test statistics : $Z = \frac{\bar{t} - 38}{5/\sqrt{n}}$

(ii) To reject H_0,
$$\frac{\bar{t}-38}{5/\sqrt{n}} < \text{invN}(5\%).$$
Given $n = 50$,
$$\frac{\bar{t}-38}{5/\sqrt{50}} < -1.64485$$
$$\bar{t} < 36.8369$$
$$0 < \bar{t} < 36.8 \quad (\text{corr to 3 sig fig}).$$
The set of values of $\bar{t} = \{\bar{t} \in \mathbb{R} \mid 0 < \bar{t} < 36.8\}$

(iii). Given that $\bar{t} = 37.1$
For the null hypothesis not to be rejected:
$$\frac{\bar{t}-38}{5/\sqrt{n}} > \text{invN}(5\%)$$
$$\frac{37.1-38}{5/\sqrt{n}} > -1.64485$$
$$\frac{-0.9\sqrt{n}}{5} > -1.64485.$$
$$\sqrt{n} < 1.6448\left(\frac{5}{0.9}\right)$$
$$\sqrt{n} < 9.1377$$
$$n < 83.4989$$

$$\therefore \quad 0 < n \leq 83 \ \#.$$

The set of values of $n = \{n \in \mathbb{Z}^+ / 0 < n \leq 83\}_\#$.

2011 H2 Maths P2:

11 Let R be No. of women.

(i) $P(R=4) = \dfrac{{}^{18}C_4 \cdot {}^{12}C_6}{{}^{30}C_{10}} = 0.0941$ (corr to 3 sig fig)

(ii). $P(R=r) > P(R=r+1)$.

$$\frac{{}^{18}C_r \;\; {}^{12}C_{10-r}}{{}^{30}C_{10}} > \frac{{}^{18}C_{r+1} \;\; {}^{12}C_{10-(r+1)}}{{}^{30}C_{10}}$$

$$\frac{\cancel{18!}}{r!\,(18-r)!}\;\frac{\cancel{12!}}{(10-r)!\,[12-(10-r)]!} > \frac{\cancel{18!}}{(r+1)!\,[18-(r+1)]!}\;\frac{\cancel{12!}}{(9-r)!\,[12-(9-r)]!}$$

$$\frac{1}{r!\,(18-r)!\,(10-r)!\,(r+2)!} > \frac{1}{(r+1)!\,(17-r)!\,(9-r)!\,(r+3)!}$$

$$(r+1)!\,(17-r)!\,(9-r)!\,(r+3)! > r!\,(18-r)!\,(10-r)!\,(r+2)!$$

(shown).

Continue from above.

$$(r+1)\cancel{r!}\;\cancel{(17-r)!}\;\cancel{(9-r)!}\;(r+3)\cancel{(r+2)!} > \cancel{r!}\,(18-r)\cancel{(17-r)!}\,(10-r)\cancel{(9-r)!}\;\cancel{(r+2)!}$$

$$(r+1)(r+3) > (18-r)(10-r).$$

$$\cancel{r^2}+4r+3 > 180-28r+\cancel{r^2}.$$

$$32r > 177$$

$$r > \frac{177}{32} = 5.53125$$

It implies we can take $r = 6, 7, 8, 9, 10$.

For $r=6$, $P(R=6) > P(R=7) > P(R=8) > P(R=9) > P(R=10)$

$\therefore$ The value of r (most probable number of women) $= 6$ #

2011 H2 Maths P2 :

12(i). Let r.V X_1 denotes no of people joining the queue in the period of 1 min.

$X_1 \sim Po(1.2)$.

Let r.V X_4 denotes no of people joining the queue. in the period of 4 mins.

$X_4 \sim Po(1.2 \times 4) = Po(4.8)$

$$P(X_4 \geq 8) = 1 - P(X_4 \leq 7)$$
$$= 1 - \text{poissoncdf}(4.8, 7)$$
$$= 0.11333$$
$$\approx 0.113 \quad \text{((cor to 3 sig fig)}$$

(ii) Let T be r.v for the no. of people joining the queue in t sec.

Note 1 min — 1.2.
60 s — 1.2
t s — $\frac{1.2}{60}t$.
$= 0.02t$.

$T \sim Po(0.02t)$.

Given $P(T \leq 1) = 0.7$.

$P(T=0) + P(T=1) = 0.7$.

$$\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} = 0.7.$$

$$e^{-\lambda}[1 + \lambda] = 0.7.$$
$$e^{-0.02t}[1 + 0.02t] = 0.7.$$

From G.C, $t = 54.867$

≈ 55 sec (cor to nearest whole nos).

2011 H2 Maths P2 :

12(iii) Let r.v X_{15} denotes the no of ppl joining the queue in 15 mins

$X_{15} \sim Po(1.2 \times 15) = Po(18)$

Since $\lambda > 10$,

$X_{15} \sim N(18, 18)$ approximately.

Let r.v. Y_{15} denote the number of ppl leaving the queue in 15 mins.

$Y_{15} \sim Po(1.8 \times 15) = Po(27)$

Since $\lambda > 10$,

$Y_{15} \sim N(27, 27)$. approximately.

$E(Y_{15} - X_{15}) = 27 - 18 = 9$.

$Var(Y_{15} - X_{15}) = Var(Y_{15}) + Var(X_{15})$

$= 27 + 18$.

$= 45$.

Note : $Y_{15} - X_{15}$ denotes the no. of ppl left the queue in 15 mins.

$\therefore Y_{15} - X_{15} \sim N(9, 45)$ approximately.

P(At least 24 people in the queue at 0945)

= P(at most 11 people left the queue, given 35 ppl in the beginning) ie. 0930

$= P(Y_{15} - X_{15} \leq 11)$

$= 0.61720$

≈ 0.617 (corr. 3 sig. fig)

(iv) The number of people joining and leaving the queue would no longer be completely random within this longer time interval of several hours, and be independent of the previous time interval since it will likely follow the flight schedule at the airport.