2011 H2 Maths P1 V1.1

1 Given $\frac{x^2+x+1}{x^2+x-2} < 0$. ——— ①

Note $x^2+x+1 = x^2+x+(\frac{1}{2})^2-(\frac{1}{2})^2+1$
$= (x+\frac{1}{2})^2+\frac{3}{4} > 0$, Since $(x+\frac{1}{2})^2 \geq 0$

From ①, hence $x^2+x-2 < 0$.
$(x+2)(x-1) < 0$.
$\therefore -2 < x < 1$ #

2. Given $f(x) = ax^2+bx+c$, where a, b, and c are constants.

(i) Given $y=4.5$ when $x=-1.5$,
$a(-1.5)^2 - 1.5b + c = 4.5$.
$2.25a - 1.5b + c = 4.5$ ——— ①.

Given $y=3.2$ when $x=2.1$.
$a(2.1)^2 + b(2.1) + c = 3.2$.
$4.41a + 2.1b + c = 3.2$ ——— ②.

Given $y=4.1$ when $x=3.4$.
$a(3.4)^2 + b(3.4) + c = 4.1$.
$11.56a + 3.4b + c = 4.1$. ——— ③.

Using G.C, we have. $a = 0.2149 \approx 0.215$ (corr to 3 dec pl)
$b = -0.4901 \approx -0.490$ (corr to 3 dec pl)
$c = 3.2811 \approx 3.281$ (corr to 3 dec pl)

(ii) $f'(x) = 2ax+b$
For $f(x)$ to be increasing function:
$f'(x) > 0$.
$2ax+b > 0$.
$x > -\frac{b}{2a} = \frac{-(-0.4901)}{2(0.2149)} = 1.1402$
$\therefore x > 1.14$ (corr to 3 sig fig)
$\therefore$ The set of values of $x = \{x \in \mathbb{R} \mid x > 1.14\}$ #

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3. Given $x = t^2$, $y = \frac{2}{t}$

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = -\frac{2}{t^2}$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = -\frac{2}{t^2} \div 2t. = -\frac{1}{t^3}$

(i) Given $y = \frac{2}{p} \Rightarrow t = p.$

when $t=p$, gradient of tangent $= \frac{dy}{dx} = -\frac{1}{p^3}$

$\therefore$ Equation of tangent:

$$y - \frac{2}{p} = -\frac{1}{p^3}(x - p^2).$$
$$y - \frac{2}{p} = -\frac{1}{p^3}x + \frac{1}{p}$$
$$y = -\frac{1}{p^3}x + \frac{3}{p} \ \# . \quad \text{——} \ (1).$$

(ii) sub $x = 0$ into (1), $y = \frac{3}{p}$.

$\therefore R(0, \frac{3}{p})$ where tangent meet y-axis.

sub $y = 0$ into (1),

$$0 = -\frac{1}{p^3}x + \frac{3}{p}.$$
$$\frac{1}{p^3}x = \frac{3}{p}$$
$$x = 3p^2.$$

$\therefore Q(3p^2, 0)$ where tangent meet x-axis.

(iii) Let M be the midpoint of QR.

$$M = \left(\frac{0+3p^2}{2}, \frac{\frac{3}{p}+0}{2}\right)$$
$$= \left(\frac{3}{2}p^2, \frac{3}{2p}\right)$$

$x = \frac{3}{2}p^2$, $y = \frac{3}{2p}$

$\therefore$ As p varies, midpoint of M:

$x = \frac{3}{2}p^2$ and $y = \frac{3}{2p}.$

$p = \frac{3}{2y}.$

$\therefore$ Cartesian equation of the locus of M:

$$x = \frac{3}{2}\left(\frac{3}{2y}\right)^2.$$
$$x = \frac{27}{8y^2}.$$
$$8xy^2 = 27 \ \#$$

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4. (i) $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots$ {using MF15}.

Given $g(x) = \cos^6 x$

$$= \left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots\right]^6$$
$$= \left[1 - \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \dots\right)\right]^6.$$
$$= 1 - \binom{6}{1}\left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \dots\right) + \binom{6}{2}\left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \dots\right)^2 - \dots$$
$$= 1 - 3x^2 + \frac{1}{4}x^4 + 15\left(\frac{1}{4}x^4 + \dots\right).$$
$$\sim 1 - 3x^2 + 4x^4 \quad \#.$$

(ii) (a) $\int_0^a g(x) = \int_0^a (1 - 3x^2 + 4x^4).\, dx$

$$= \left[x - x^3 + \frac{4}{5}x^5\right]_0^a$$
$$= a - a^3 + \frac{4}{5}a^5.$$

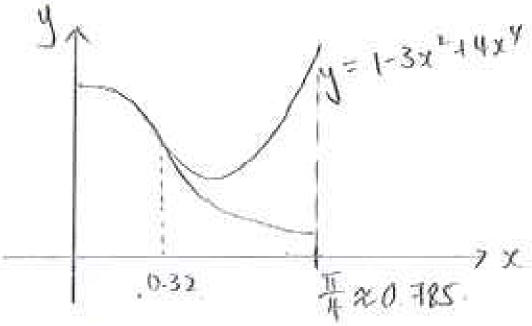
when $a = \frac{\pi}{4}$, $\int_0^{\pi/4} g(x) = \frac{\pi}{4} - \frac{\pi^3}{64} + \frac{1}{1280}\pi^5$

$$\approx 0.540 \quad \text{(corr to 3 sig fig)}.$$

(ii) (b) $\int_0^{\pi/4} g(x) = \int_0^{\pi/4} \cos^6 x \, dx$

$$= 0.4746$$
$$\approx 0.475 \quad \text{(corr to 3 sig fig)}.$$

Using G.C. $y = 1 - 3x^2 + 4x^4$
and $y = \cos^6 x$
we have.



From the diagram above:
the graph of $y = 1 - 3x^2 + 4x^4$ differs significantly from the graph of $y = \cos^6 x$ for values of x beyond 0.32.
$\therefore$ The approximation of $\int_0^a g(x)\, dx$ in part (ii) is not very good since the upper limit $a = \frac{\pi}{4} > 0.32$.

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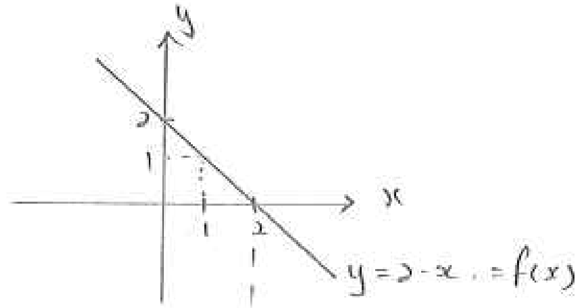
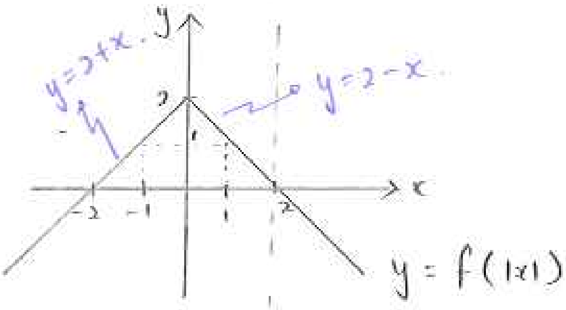
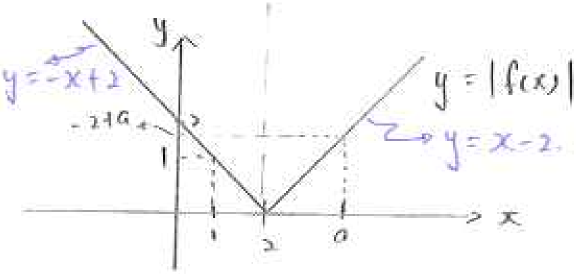
5 $f(x) = 2 - x$.



(i) $y = f(|x|)$



$y = |f(x)|$



(ii) For $f(|x|) = |f(x)|$, we observe from the graphs:
$0 \leq x \leq 2$.

$\therefore$ Set of values of x is $\{ x \in \mathbb{R} \mid 0 \leq x \leq 2 \}$ #

(iii) $\int_{-1}^{1} f(|x|)\,dx = 2 \times$ Area of trapz.
$= 2 \times \frac{1}{2}(1)[1+2]$
$= 3$ units^2

$\int_{1}^{a} |f(x)|\,dx$ = Area of Δ from $x=1$ to $x=2$ + Area of Δ from $x=1$ to $x=a$
$= \frac{1}{2}(1)(1) + \frac{1}{2}(a-2)(-2+a)$.
$= \frac{1}{2} + \frac{1}{2}(a-2)^2$. units^2

Hence $\int_{-1}^{1} f(|x|)\,dx = \int_{1}^{a} |f(x)|\,dx$.
$3 = \frac{1}{2} + \frac{1}{2}(a-2)^2$.
$6 = 1 + (a-2)^2$.
$(a-2)^2 = 5$.
$\therefore a = 2 + \sqrt{5}$ # or $a = 2 - \sqrt{5}$ (reject since a lies on +ve x-axis).

$a = 2 + \sqrt{5}$ #

5 (iii) Alternative method:

$$\int_{-1}^{1} f(|x|) = 2\int_0^1 f(x)\,dx.$$
$$= 2\int_0^1 (2-x)\,dx.$$
$$= 2\left[2x - \frac{x^2}{2}\right]_0^1$$
$$= 2\left[2 - \frac{1}{2}\right]$$
$$= 3 \text{ units}^2.$$

$$\int_{-1}^{1} f(|x|)\,dx = \int_1^a |f(x)|\,dx.$$
$$3 = \int_1^2 f(x)\,dx + \int_2^a - f(x)\,dx.$$
$$3 = \int_1^2 (2-x)\,dx + \int_2^a (x-2)\,dx.$$
$$3 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^a$$
$$3 = \left[(4-2)-\left(2-\frac{1}{2}\right)\right] + \left[\frac{a^2}{2} - 2a - (2-4)\right]$$
$$3 = \frac{1}{2} + \frac{a^2}{2} - 2a + 2.$$
$$6 = 1 + a^2 - 4a + 4.$$
$$a^2 - 4a + 1 = 0$$
$$a = \frac{4 \pm \sqrt{20}}{2}$$
$$= \frac{4 \pm 2\sqrt{5}}{2}$$
$$= 2+\sqrt{5} \quad \text{or} \quad 2-\sqrt{5} \ \text{(Not applicable)}.$$

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6(i). Need to prove $\sin(r+\frac{1}{2})\theta - \sin(r-\frac{1}{2})\theta \equiv 2\cos r\theta \sin\frac{1}{2}\theta$.

$$\begin{aligned}
LHS &= \sin(r+\tfrac{1}{2})\theta - \sin(r-\tfrac{1}{2})\theta \\
&= \sin(r\theta + \tfrac{1}{2}\theta) - \sin(r\theta - \tfrac{1}{2}\theta). \\
&= \sin r\theta \cos\tfrac{1}{2}\theta + \cos r\theta \sin\tfrac{1}{2}\theta - [\sin r\theta \cos\tfrac{1}{2}\theta - \cos r\theta \sin\tfrac{1}{2}\theta]. \\
&= \sin r\theta \cos\tfrac{1}{2}\theta + \cos r\theta \sin\tfrac{1}{2}\theta - \sin r\theta \cos\tfrac{1}{2}\theta + \cos r\theta \sin\tfrac{1}{2}\theta. \\
&= 2\cos r\theta \sin\tfrac{1}{2}\theta \\
&= RHS \text{ (Proved)}_{\#}.
\end{aligned}$$

(ii) from 6(i):

$$\sin(r+\tfrac{1}{2})\theta - \sin(r-\tfrac{1}{2})\theta = 2\cos r\theta \sin\tfrac{1}{2}\theta.$$

taking $\sum_{r=1}^{n}$ on both sides:

$$\begin{aligned}
\sum_{r=1}^{n}[\sin(r+\tfrac{1}{2})\theta - \sin(r-\tfrac{1}{2})\theta] &= \sum_{r=1}^{n} 2\cos r\theta \sin\tfrac{1}{2}\theta. \\
2\sin\tfrac{1}{2}\theta \sum_{r=1}^{n}\cos r\theta &= \sum_{r=1}^{n}[\sin(r+\tfrac{1}{2})\theta - \sin(r-\tfrac{1}{2})\theta] \\
\therefore \sum_{r=1}^{n}\cos r\theta &= \frac{1}{2\sin\frac{1}{2}\theta}\sum_{r=1}^{n}[\sin(r+\tfrac{1}{2})\theta - \sin(r-\tfrac{1}{2})\theta] \\
&= \frac{1}{2\sin\frac{1}{2}\theta}\Big[\sin\tfrac{3}{2}\theta - \sin\tfrac{1}{2}\theta \\
&\qquad + \sin\tfrac{5}{2}\theta - \sin\tfrac{3}{2}\theta \\
&\qquad + \sin\tfrac{7}{2}\theta - \sin\tfrac{5}{2}\theta \\
&\qquad \vdots \\
&\qquad + \sin(n-\tfrac{1}{2})\theta - \sin(n-\tfrac{3}{2})\theta \\
&\qquad + \sin(n+\tfrac{1}{2})\theta - \sin(n-\tfrac{1}{2})\theta\Big] \\
&= \frac{1}{2\sin\frac{1}{2}\theta}[\sin(n+\tfrac{1}{2})\theta - \sin\tfrac{1}{2}\theta].
\end{aligned}$$

6 (iii). Let $P(n)$ denote the statement s.t.

$$\sum_{r=1}^{n} \sin r\theta = \frac{\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}, \quad \forall n \in \mathbb{Z}^+$$

When $n=1$

$$LHS = \sum_{r=1}^{1} \sin r\theta = \sin(1)\theta = \sin\theta.$$

$$RHS = \frac{1}{2\sin\frac{1}{2}\theta}\left[\cos\frac{1}{2}\theta - \cos\frac{3}{2}\theta\right].$$

$$= \frac{1}{2\sin\frac{1}{2}\theta}\left[-2\sin\frac{\frac{1}{2}\theta+\frac{3}{2}\theta}{2}\sin\frac{\frac{1}{2}\theta-\frac{3}{2}\theta}{2}\right]$$ Using $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$$= \frac{1}{2\sin\frac{1}{2}\theta}\left[-2\sin\theta\sin\left(-\frac{\theta}{2}\right)\right].$$ Using $\sin(-\theta) = -\sin\theta$.

$$= \sin\theta.$$

$\therefore$ LHS = R.H.S.

$\therefore$ P_1 is true.

Assum P_k is true. i.e.

$$\sum_{r=1}^{k} \sin r\theta = \frac{\cos\frac{1}{2}\theta - \cos(k+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}, \text{ for some } k \in \mathbb{Z}^+$$

To show that P_{k+1} is also true.

i.e. $$\sum_{r=1}^{k+1} \sin r\theta = \frac{\cos\frac{1}{2}\theta - \cos(k+\frac{3}{2})\theta}{2\sin\frac{1}{2}\theta}.$$

$$L.H.S = \sum_{r=1}^{k+1} \sin r\theta$$

$$= \sum_{r=1}^{k} \sin r\theta + \sin(k+1)\theta.$$

$$= \frac{\cos\frac{1}{2}\theta - \cos(k+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} + \sin(k+1)\theta.$$

$$= \frac{1}{2\sin\frac{1}{2}\theta}\left[\cos\frac{1}{2}\theta - \cos(k+\frac{1}{2})\theta + 2\sin\frac{1}{2}\theta\sin(k+1)\theta\right].$$

Note: $2\sin A\sin B = \cos(A-B) - \cos(A+B)$.

$$2\sin\frac{1}{2}\theta\sin(k+1)\theta = \cos\left[\frac{1}{2}\theta - (k+1)\theta\right] - \cos\left[\frac{1}{2}\theta + (k+1)\theta\right]$$

$$= \cos\left[-(k+\frac{1}{2})\theta\right] - \cos\left[(k+\frac{3}{2})\theta\right]$$

$$= \cos(k+\frac{1}{2})\theta - \cos(k+\frac{3}{2})\theta$$

$$= \frac{1}{2\sin\frac{1}{2}\theta}\left[\cos\frac{1}{2}\theta - \cos(k+\frac{1}{2})\theta + \cos(k+\frac{1}{2})\theta - \cos(k+\frac{3}{2})\theta\right].$$

$$= \frac{1}{2\sin\frac{1}{2}\theta}\left[\cos\frac{1}{2}\theta - \cos(k+\frac{3}{2})\theta\right].$$

$$= RHS.$$

$\therefore$ $P(k+1)$ is true if $P(k)$ is true.

Since P_1 is true and $P(k+1)$ is true if $P(k)$ is true, by mathematical induction, $P(n)$ is true, $\forall n \in \mathbb{Z}^+$

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7 Given $\vec{OA} = \underset{\sim}{a}$ and $\vec{OB} = \underset{\sim}{b}$

Given $\frac{OP}{PA} = \frac{1}{2}$.

$\frac{OP}{OA} = \frac{1}{3}$. $\therefore \vec{OP} = \frac{1}{3}\vec{OA} = \frac{1}{3}\underset{\sim}{a}$.

Given $\frac{OQ}{QB} = \frac{3}{2}$.

$\frac{OQ}{OB} = \frac{3}{5}$ $\therefore \vec{OQ} = \frac{3}{5}\vec{OB} = \frac{3}{5}\underset{\sim}{b}$.

(i) Given M is the midpoint of PQ,

by ratio thm. $\vec{OM} = \frac{1}{2}[\vec{OP} + OQ]$

$= \frac{1}{2}[\frac{1}{3}a + \frac{3}{5}\underset{\sim}{b}]$.

$= \frac{1}{30}[5\underset{\sim}{a} + 9\underset{\sim}{b}]$

Area of $\Delta OMP = \frac{1}{2}|\vec{OM}||\vec{OP}|\sin\theta$.

$= \frac{1}{2}|\vec{OM} \times \vec{OP}|$

$= \frac{1}{2}|\frac{1}{30}(5\underset{\sim}{a}+9\underset{\sim}{b}) \times \frac{1}{3}\underset{\sim}{a}|$

$= \frac{1}{180}|(5\underset{\sim}{a}+9\underset{\sim}{b}) \times \underset{\sim}{a}|$

$= \frac{1}{180}|(5\underset{\sim}{a}\times\underset{\sim}{a} + 9\underset{\sim}{b}\times\underset{\sim}{a})|$

$= \frac{1}{180}|\underset{\sim}{0} + 9\underset{\sim}{b}\times\underset{\sim}{a}|$ $\{\underset{\sim}{a}\times\underset{\sim}{a} = \underset{\sim}{0}\}$

$= \frac{1}{180}9|\underset{\sim}{b}\times\underset{\sim}{a}|$

$= \frac{1}{20}|\underset{\sim}{a}\times\underset{\sim}{b}|$ $\{|\underset{\sim}{a}\times\underset{\sim}{b}| = |\underset{\sim}{b}\times\underset{\sim}{a}|\}$

$\therefore k = \frac{1}{20}$.

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7 (ii) Given $\underset{\sim}{a} = \begin{pmatrix} 2p \\ -6p \\ 3p \end{pmatrix}$ $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, p is +ve constant.

(a) Given $\underset{\sim}{a}$ is a unit vector.

Then $|\underset{\sim}{a}| = 1$.

$(2p)^2 + (-6p)^2 + (3p)^2 = 1^2$.

$4p^2 + 36p^2 + 9p^2 = 1$

$p^2 = \frac{1}{49}$.

$p = \frac{1}{7}$ or $p = -\frac{1}{7}$ (reject cos p is +ve).

$\therefore p = \frac{1}{7}$ #.

(b). $|\underset{\sim}{a} \cdot \underset{\sim}{b}| = |\,|a||b| \cos\theta\,|$

$= |b|\,|\cos\theta|$

Since $\underset{\sim}{a}$ is a unit vector, $|a \cdot b|$ is the length of projection of projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ #

(c) $\underset{\sim}{a} \times \underset{\sim}{b} = \frac{1}{7}\begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$= \frac{1}{7}\left[\begin{pmatrix} 9 \\ 7 \\ 8 \end{pmatrix}\right]$

$= \frac{1}{7}\begin{pmatrix} 9 \\ 7 \\ 8 \end{pmatrix}$ #

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8 (i). $\int \frac{1}{100-v^2}\,dv = \int \frac{1}{10^2-v^2}\,dv$

$= \frac{1}{20}\ln\left|\frac{10+v}{10-v}\right| + C$ {using MF-15 formulae}.

(ii) (a) Given $\frac{dv}{dt} = 10 - 0.1v^2$.

$\frac{dv}{dt} = \frac{1}{10}\left[100 - v^2\right]$.

$\int \frac{1}{100-v^2}\,dv = \frac{1}{10}\int 1\,dt$.

$\frac{1}{20}\ln\left|\frac{10+v}{10-v}\right| = \frac{1}{10}t + C$, where C is a constant.

$\frac{1}{10}t = \frac{1}{20}\ln\left|\frac{10+v}{10-v}\right| - C$.

$t = \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right| + C_1$, where C_1 is a constant.

$v = 0$ when $t = 0$.

$0 = \frac{1}{2}\ln\left|\frac{10}{10}\right| + C_1$

$\therefore C_1 = 0$.

$\therefore t = \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|$

when $v = 5$ m/s,

$t = \frac{1}{2}\ln\left|\frac{10+5}{10-5}\right| = \frac{1}{2}\ln 3$ #

(ii) (b). when $t = 1$,

$1 = \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|$

$2 = \ln\left|\frac{10+v}{10-v}\right|$

$10 + v = e^2(10 - v)$.

$10 + v = 10e^2 - e^2 v$.

$(1+e^2)v = 10(e^2-1)$.

$v = \frac{10(e^2-1)}{(e^2+1)} \approx 7.62\text{ ms}^{-1}$ # (corr. to 3 sig fig)

(c) Since $t = \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|$.

$e^{2t} = \frac{10+v}{10-v}$.

$10e^{2t} - e^{2t}v = 10 + v$.

$10e^{2t} - 10 = (1+e^{2t})v$.

$v = \frac{10(e^{2t}-1)}{e^{2t}+1}$

$v = \frac{10(1-e^{-2t})}{1+e^{-2t}}$.

when $t \to \infty$, $e^{-2t} \to 0$.

$\therefore v \approx \frac{10(1-0)}{1+0}$

$= 10\text{ ms}^{-1}$ #

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9(i). Machine A.

Day	1	2	3
Drilled depth.	256	256-7	256-2(7)

$a = 256. \quad d = -7.$

Using A.P :

$T_{10} = 256 + (10-1)(-7).$

$= 256 + 9(-7)$

$= 193 \text{ m}.$

Let n be the day when the drilled depth is less than 10m.

$T_n < 10.$

$256 + (n-1)(-7) < 10.$

$246 < 7(n-1).$

$n > 36\frac{1}{7}.$

$\therefore$ 37th day will be the day when the drilled depth is less than 10 m.

$\therefore$ The total depth for 37 days

$= S_{37}$

$= \frac{37}{2}\left[2(256) + (37-1)(-7)\right]$

$= 4810 \text{ m}$ #

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9 (ii) Machine B :

Day	1	2	3	4
Drilled depth	256	$\left(\frac{8}{9}\right)(256)$	$\left(\frac{8}{9}\right)^2(256)$	$\left(\frac{8}{9}\right)^3(256)$

$a = 256$, $r = \frac{8}{9}$.

By G.P.

Theoretical max total depth $= S_\infty = \frac{a}{1-r}$

$= \frac{256}{1-\frac{8}{9}}$

$= 2304$ m.

Let m be the no of days to exceed 99% of the theoretical max total depth.

$$S_m > \frac{99}{100} S_\infty .$$

$$\frac{256\left[1-\left(\frac{8}{9}\right)^m\right]}{1-\frac{8}{9}} > \frac{99}{100} \times 2304 .$$

$$2304\left[1-\left(\frac{8}{9}\right)^m\right] > \frac{99}{100} \times 2304 .$$

$$1-\left(\frac{8}{9}\right)^m > 0.99 .$$

$$0.01 > \left(\frac{8}{9}\right)^m .$$

$$m \lg\left(\frac{8}{9}\right) < \lg 0.01$$

$$m > \lg 0.01 / \lg\left(\frac{8}{9}\right) . = 39.098$$

$\therefore$ m = 40 . days #.

10 (i). Given $z^2 = -8i$.

Let $z = a + ib$, where $a, b \in \mathbb{R}$

$$(a+ib)^2 = -8i.$$

$$a^2 - b^2 + 2abi = 0 - 8i.$$

Then we have (by comparing)

$$a^2 - b^2 = 0. \quad \text{——} ①$$

$$2ab = -8$$

$$ab = -4$$

$$b = -\frac{4}{a} \quad \text{——} ②.$$

Sub ② into ①.

$$a^2 = \frac{16}{a^2}.$$

$$a^4 = 16.$$

$a^2 = 4$ or $a^2 = -4$ (Not applicable).

$a = \pm 2$.

when $a = 2$, $b = -2$. (from ②)

when $a = -2$, $b = 2$ (from ②).

$\therefore$ $z_1 = 2 - 2i$. and $z_2 = -2 + 2i$. #.

(ii). Given

$$w^2 + 4w + (4 + 2i) = 0.$$

$$w^2 + 4w + 4 = -2i.$$

$$\therefore (w+2)^2 = -2i.$$

$$4(w+2)^2 = -8i.$$

$$(2w+4)^2 = -8i.$$

Using part (i) result, we have:

$2w_1 + 4 = 2 - 2i$. or $2w_2 + 4 = -2 + 2i$.

$2w_1 = -2 - 2i$ $\quad$ $2w_2 = -6 + 2i$.

$w_1 = -1 - i$. $\quad$ $w_2 = -3 + i$.

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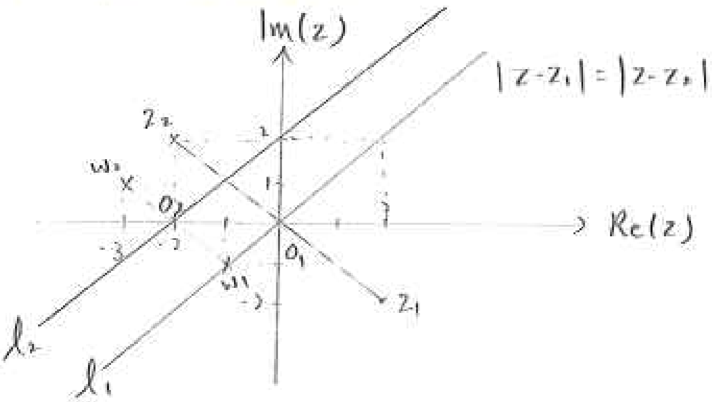
10 (iii) (a) $z_1 = 2-2i.$ $\quad z_2 = -2+2i.$

$w_1 = -1-i.$ $\quad w_2 = -3+i.$

$|z - z_1| = |z - z_2|$

$|z - (2-2i)| = |z - (-2+2i)|$



(iv). $z_1(2,-2)$, $z_2(-2,2)$.

Gradient of $z_1 z_2 = \frac{-2-2}{2+2} = -1.$

Gradient of $l_1 = -\frac{1}{-1} = 1.$

$w_1(-1,-1)$, $w_2(-3,1)$.

Gradient of $w_1 w_2 = \frac{-1-1}{-1+3} = \frac{-2}{2} = -1.$

Gradient of $l_2 = -\frac{1}{-1} = 1.$

Since gradient of l_1 = gradient of l_2,
there are no points which lie on both of these loci.

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11 Let A be $(4,-1,-3)$ $\quad \vec{OA} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$.

Let B be $(-2,-5,2)$ $\quad \vec{OB} = \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}$

Let C be $(4,-3,-2)$ $\quad \vec{OC} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}$

$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} -6 \\ -4 \\ 5 \end{pmatrix}$.

$\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

$\vec{AB} \times \vec{AC} = \begin{pmatrix} -6 \\ -4 \\ 5 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Since $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is $\perp$ to plane p

Let $\underset{\sim}{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

$\therefore$ Vector equation of plane p :

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}.$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = -3.$$

Let $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = -3.$$

$$x + y + 2z = -3$$ (Cartesian equation)$_{\#}$.

11 (ii) Given l_1 : $\frac{x-1}{2} = \frac{y-2}{-4} = \frac{z+3}{1}$

Then

$$\underset{\sim}{r} = \begin{pmatrix}1\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}2\\-4\\1\end{pmatrix} \quad \text{—① , where } \lambda \in \mathbb{R}.$$

Given l_2 : $\frac{x+2}{1} = \frac{y-1}{5} = \frac{z-3}{k}$

Then

$$\underset{\sim}{r} = \begin{pmatrix}-2\\1\\3\end{pmatrix} + \mu\begin{pmatrix}1\\5\\k\end{pmatrix} \quad \text{—② , where } \mu \in \mathbb{R}.$$

Given that l_1 and l_2 intersect.

① = ②.

$$\begin{pmatrix}1\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}2\\-4\\1\end{pmatrix} = \begin{pmatrix}-2\\1\\3\end{pmatrix} + \mu\begin{pmatrix}1\\5\\k\end{pmatrix}. \quad \text{——— (A).}$$

$$\Rightarrow \quad 1+2\lambda = -2+\mu.$$

$$\mu = 3+2\lambda \quad \text{——— ③.}$$

$$2-4\lambda = 1+5\mu$$

$$4\lambda+5\mu = 1 \quad \text{——— ④}$$

sub ③ into ④,

$$4\lambda + 5(3+2\lambda) = 1$$

$$4\lambda + 15 + 10\lambda = 1.$$

$$14\lambda = -14.$$

$$\lambda = -1.$$

Put $\lambda = -1$ into ③,

$$\mu = 3+2(-1)$$

$$\mu = 1.$$

From (A), we also have.

$$-3+\lambda = 3+\mu k.$$

Since l_1 and l_2 intersect,

$$-3+(-1) = 3+(1)k.$$

$$-4 = 3+k.$$

$$k = -7 \#.$$

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11 (iii). $l_1 : \quad \underset{\sim}{r} = \begin{pmatrix}1\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}$. , Let $\underset{\sim}{d} = \begin{pmatrix}2\\-4\\1\end{pmatrix}$, $\underset{\sim}{a} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$.

$p : \quad \underset{\sim}{r} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} = -3$. —(A) Let $\underset{\sim}{n} = \begin{pmatrix}1\\1\\2\end{pmatrix}$.

$$\begin{aligned}\underset{\sim}{d} \cdot \underset{\sim}{n} &= \begin{pmatrix}2\\-4\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} \\ &= 2 + (-4) + 2 \\ &= 0.\end{aligned}$$

$\underset{\sim}{d} \perp \underset{\sim}{n}$.
⇒ line l_1 is parallel to plane p. —(B)

$$\begin{aligned}\underset{\sim}{a} \cdot \underset{\sim}{n} &= \begin{pmatrix}1\\2\\-3\end{pmatrix} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} \\ &= 1 + 2 - 6 \\ &= -3.\end{aligned}$$

⇒ $\underset{\sim}{a}$ lies on plane p. —(C).
From (B) and (C), l_1 lies in p #.

$l_2 : \quad \underset{\sim}{r} = \begin{pmatrix}-2\\1\\3\end{pmatrix} + \mu\begin{pmatrix}1\\5\\-7\end{pmatrix}$, $d_2 = \begin{pmatrix}1\\5\\-7\end{pmatrix}$

$\underset{\sim}{r} = \begin{pmatrix}-2+\mu\\1+5\mu\\3-7\mu\end{pmatrix}$

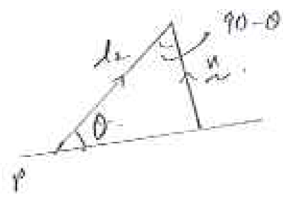
$\begin{pmatrix}-2+\mu\\1+5\mu\\3-7\mu\end{pmatrix} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} = -3$. {using (A)}

$$\begin{aligned}-2+\mu+1+5\mu+2(3-7\mu) &= -3. \\ -2+\mu+1+5\mu+6-14\mu &= -3. \\ 8\mu &= 8 \\ \mu &= 1.\end{aligned}$$

$\underset{\sim}{r} = \begin{pmatrix}-2+1\\1+5\\3-7\end{pmatrix} = \begin{pmatrix}-1\\6\\-4\end{pmatrix}$

∴ Coordinates of the point at which l_2 intersect p = $(-1, 6, -4)$.

(iv) Let θ be the acute angle between l_2 and p.



$$\begin{aligned}|\underset{\sim}{n} \cdot d_2| &= |\underset{\sim}{n}|\,|\underset{\sim}{d_2}| \cos(90° - \theta). \\ \left|\begin{pmatrix}1\\1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\5\\-7\end{pmatrix}\right| &= \sqrt{1+1+4}\,\sqrt{1+25+49}\,\sin\theta. \\ \sin\theta &= \frac{|1+5-14|}{\sqrt{6}\,\sqrt{75}} \\ \sin\theta &= \frac{8}{\sqrt{450}}\end{aligned}$$

$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{8}{\sqrt{450}}\right) \approx 22.2°$ (corr to 1 dec pl)