MATHEMATICS (H2) 9740/01
Paper 1 Suggested Solutions — October/November 2010

1. **Topic: Vectorsin Three Dimensions(Points & Lines)**

(i) Given $\mathbf{a} = \begin{pmatrix}2p\\3p\\6p\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$, where $p > 0$.

$|\mathbf{a}| = p\sqrt{2^2+3^2+6^2} = 7p$

$|\mathbf{b}| = \sqrt{1^2+(-2)^2+(2)^2} = 3$

Since $|\mathbf{a}| = |\mathbf{b}|$

$7p = 3$

$\therefore \boldsymbol{p} = \frac{3}{7}$

(ii) $\mathbf{a} = \frac{3}{7}\begin{pmatrix}2\\3\\6\end{pmatrix} = \begin{pmatrix}\frac{6}{7}\\\frac{9}{7}\\\frac{18}{7}\end{pmatrix}$

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \left[\begin{pmatrix}\frac{6}{7}\\\frac{9}{7}\\\frac{18}{7}\end{pmatrix}+\begin{pmatrix}1\\-2\\2\end{pmatrix}\right]\cdot\left[\begin{pmatrix}\frac{6}{7}\\\frac{9}{7}\\\frac{18}{7}\end{pmatrix}-\begin{pmatrix}1\\-2\\2\end{pmatrix}\right]$$

$$= \begin{pmatrix}\frac{13}{7}\\\frac{-5}{7}\\\frac{32}{7}\end{pmatrix}\cdot\begin{pmatrix}\frac{-1}{7}\\\frac{23}{7}\\\frac{4}{7}\end{pmatrix}$$

$$= -\frac{13}{49}-\frac{115}{49}+\frac{128}{49}$$

$$= 0$$

$\therefore (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = $ **0 (shown)**

ALTERNATIVE APPROACH

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{b}$$
$$= |\mathbf{a}|^2 - |\mathbf{b}|^2$$
$$= |\mathbf{a}|^2 - |\mathbf{a}|^2 \text{(since } |\mathbf{a}| = |\mathbf{b}|\text{)}$$
$$= 0$$

> $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

2. **Topic: Maclaurin's Series**

> From MF15:
> $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$
> $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \cdots$

(i) $e^x(1+\sin 2x)$

$$= \left[1 + x + \tfrac{1}{2}x^2 + \cdots\right]\left[1 + 2x - \cdots\right]$$

> Sub $2x$ into $\sin x$ expansion

$$= 1 + 2x + x + 2x^2 + \tfrac{1}{2}x^2 + \cdots$$

$$= \boldsymbol{1 + 3x + \tfrac{5}{2}x^2} + \cdots \ldots\ldots\ldots (1)$$

(ii) $(1+\frac{4}{3}x)^n$

> From MF15:
> $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

$$= 1 + \tfrac{n}{1}(\tfrac{4}{3}x)^1 + \tfrac{n(n-1)}{2!}(\tfrac{4}{3}x)^2 + \cdots$$

$$= 1 + \tfrac{4}{3}nx + \tfrac{8}{9}n(n-1)x^2 + \cdots \ldots\ldots\ldots (2)$$

Comparing coefficients of x in (1) and (2)

$\frac{4}{3}n = 3$

$\boldsymbol{n = \frac{9}{4}}$

$\therefore$ **Third term in series (2)**

$$= \tfrac{8}{9}\left(\tfrac{9}{4}\right)\left(\tfrac{9}{4} - 1\right)x^2$$

$$= \tfrac{5}{2}x^2$$

= **Third term in series (1) (shown)**

3. **Topic: Arithmetic Progression, Recurrence**

(i) Given $S_n = n(2n + c)$

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= n(2n + c) - (n-1)[2(n-1) + c] \\ &= 2n^2 + nc - 2(n-1)^2 - c(n-1) \\ &= 2n^2 + nc - 2(n^2 - 2n + 1) - cn + c \\ &= \mathbf{4n + c - 2} \end{aligned}$$

ALTERNATIVE APPROACH

$$\begin{aligned} & S_1 = u_1 = 1[2(1) + c] = 2 + c \\ & S_2 = u_2 + u_1 = 2[4 + c] = 8 + 2c \\ \Rightarrow\ & u_2 = S_2 - S_1 = 8 + 2c - 2 - c = 6 + c \\ & S_3 = u_3 + u_2 + u_1 = 3(6 + c) = 18 + 3c \\ \Rightarrow\ & u_3 = S_3 - S_2 = 18 + 3c - 8 - 2c = 10 + c \\ & S_4 = u_4 + u_3 + u_2 + u_1 = 4[8 + 2c] = 32 + 4c \\ \Rightarrow\ & u_4 = S_4 - S_3 = 32 + 4c - 18 - 3c = 14 + c \end{aligned}$$

Since $u_2 - u_1 = u_3 - u_2 = u_4 - u_3 = 4$

$\Rightarrow u_n$ is an arithmetic progression (A. P.).

> 1^{st} term, $a = 2 + c$
> Common difference, $d = 4$

$$\begin{aligned} \therefore u_n &= 2 + c + (n-1)4 \\ &= 2 + c + 4n - 4 \\ &= \mathbf{4n + c - 2} \end{aligned}$$

> <u>n^{th} term in an A.P.</u>:
> $T_n = a + (n-1)d$

(ii)
$$\begin{aligned} u_n &= 4n + c - 2 \\ u_{n+1} &= 4(n+1) + c - 2 \\ &= 4n + 4 + c - 2 \\ &= (4n + c - 2) + 4 \\ &= \mathbf{u_n + 4} \end{aligned}$$

ALTERNATIVE APPROACH

From u_n in (i):

$$\begin{aligned} u_1 &= 2 + c \\ u_2 &= 6 + c = 4 + (2 + c) = 4 + u_1 \\ u_3 &= 10 + c = 4 + (6 + c) = 4 + u_2 \\ u_4 &= 14 + c = 4 + (10 + c) = 4 + u_3 \\ \therefore \mathbf{u_{n+1}} &= \mathbf{4 + u_n} \end{aligned}$$

> $f(u_n) = 4 + u_n$

4. **Topic: Differentiation (Tangents &Normals)**

(i) Given $x^2 - y^2 + 2xy + 4 = 0$ ………………… (1)

Using implicit differentiation,

$$2x - 2y\frac{dy}{dx} + \left(2y + 2x\frac{dy}{dx}\right) = 0$$
$$x + y = (y - x)\frac{dy}{dx}$$
$$\therefore \frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\boldsymbol{x+y}}{\boldsymbol{y-x}}$$

(ii) For the tangent of the curve to be parallel to the x-axis,

$$\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{x+y}{y-x} = 0$$
$$y = -x \text{ ……………… (2)}$$

Sub (2) into (1)

$$x^2 - (-x)^2 + 2x(-x) + 4 = 0$$
$$x^2 - x^2 - 2x^2 + 4 = 0$$
$$2x^2 = 4$$
$$x = \pm\sqrt{2}$$

Sub $x = -\sqrt{2}$ into (2)

$$y = \sqrt{2}$$

Sub $x = \sqrt{2}$ into (2)

$$y = -\sqrt{2}$$

The coordinates are $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$.

5. **Topic: Functions, Graphs (Transformations)**

(i) Given $y = x^3 = \text{f}(x)$

↓ $\text{f}(x-2)$
Translate 2 units in the positive x- direction

$y = (x-2)^3 = \text{g}(x)$

↓ $\frac{1}{2}\text{g}(x)$
Stretch with scale factor $\frac{1}{2}$ parallel to the y- axis

$y = \frac{1}{2}(x-2)^3 = \text{h}(x)$

↓ $\text{h}(x) - 6$
Translate 6 units in the negative y -direction

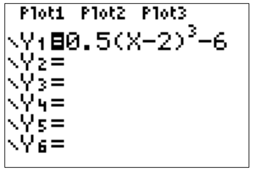
$$\boldsymbol{y = \frac{1}{2}(x-2)^3 - 6}$$

When $x = 0$, $y = \frac{1}{2}(-2)^3 - 6$
$= -10$

When $y = 0$, $0 = \frac{1}{2}(x-2)^3 - 6$
$(x-2)^3 = 12$
$x = 2 + \sqrt[3]{12}$

$\therefore$ **Curve crosses the y-axis at $(0, -10)$ and the x-axis at $(2 + \sqrt[3]{12}, 0)$.**

(ii) Using G. C. (refer to Appendix for detailed steps),



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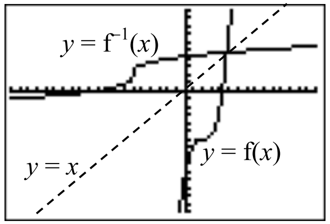
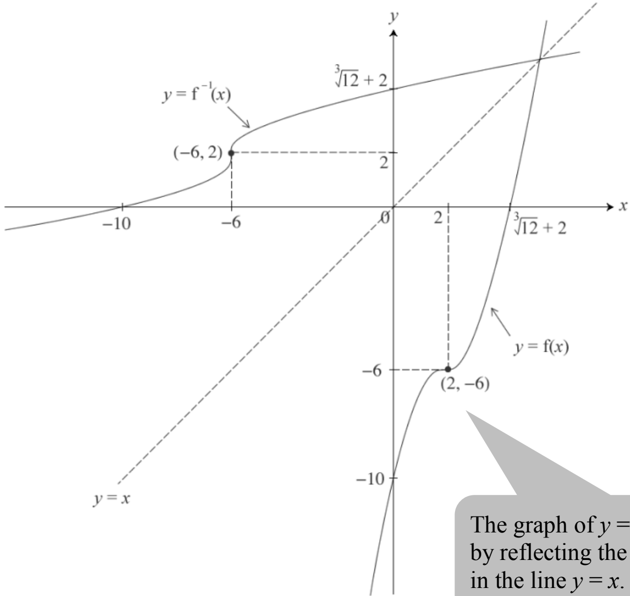




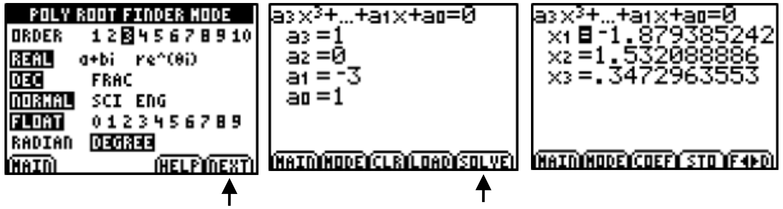
The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ in the line $y = x$.

For $y = f^{-1}(x)$, when $y = 0,\ x = -10$
$x = 0,\ y = 2 + \sqrt[3]{12}$

The x-coordinate in $f(x)$ becomes the y-coordinate in $f^{-1}(x)$ and vice versa.

6. **Topic: Integration**

(i) Using G.C. for $y = x^3 - 3x + 1$ (refer to Appendix for detailed steps and various methods that can be used),



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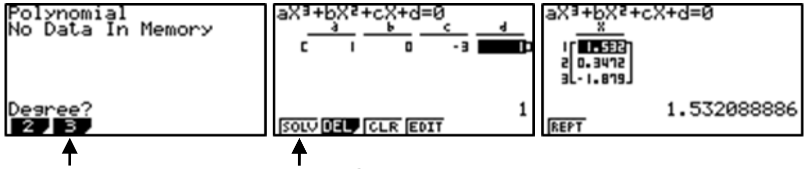

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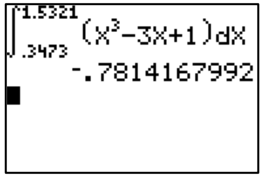
$\beta = 0.3473,\ \gamma = 1.5321$

$\therefore$ **β = 0.347 and γ = 1.532(3d.p.)**

(ii) Using G.C. (refer to Appendix for detailed steps and various methods used),



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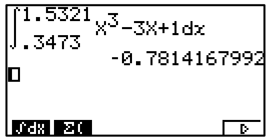

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Area ofthe region bounded by the curve and the x- axis (between $x = \beta$ and $x = \gamma$) $= \left|\int_\beta^\gamma (x^3 - 3x + 1)\,dx\right| = |(-0.7814168)|$

$\approx$ **0.781 units^2(3sig.fig.)**

Remember to obtain the absolute value for the answer as we are finding the area.

(iii) Obtain the x-coordinates of the intersection points for

$y = x^3 - 3x + 1$ ………………… (1)

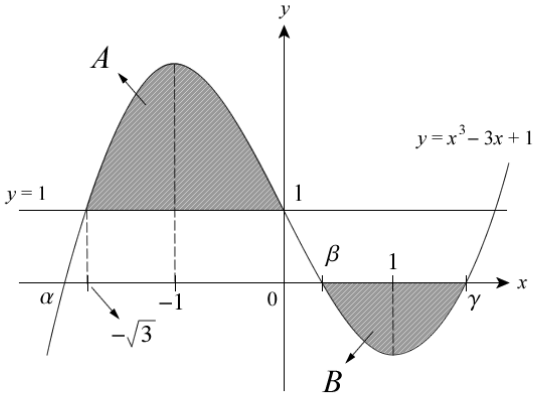
$y = 1$ ………………… (2)

(1) = (2)

$x^3 - 3x + 1 = 1$

$x(x^2 - 3) = 0$

$x = 0, x = \pm\sqrt{3}$



From the diagram, the lower and upper limit of integration to find the region is $-\sqrt{3}$ and 0 respectively.

Hence, area of region (shaded region A) between the curve and the line

$= \int_{-\sqrt{3}}^{0}(x^3 - 3x + 1)\,\mathrm{d}x - \left(1 \times \sqrt{3}\right)$

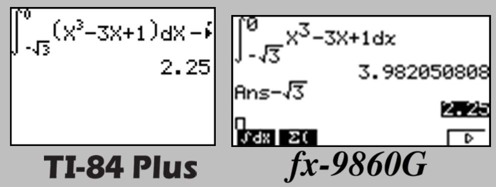
Area of rectangle

$= \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 + x\right]_{-\sqrt{3}}^{0} - \sqrt{3}$

$= \left[0 - \left(\frac{1}{4} \times 9 - \frac{3}{2}(3) - \sqrt{3}\right)\right] - \sqrt{3}$

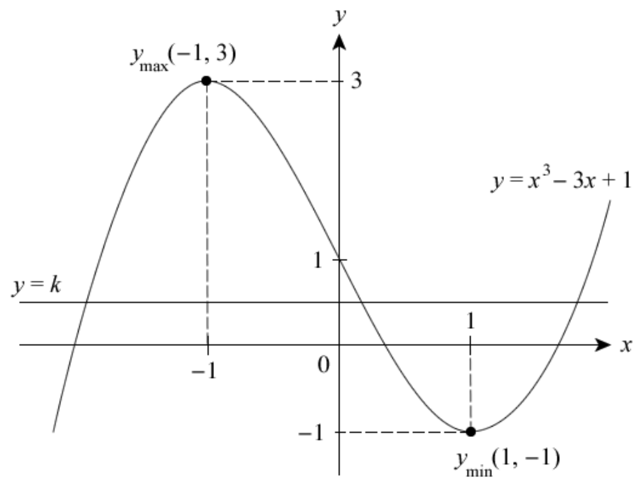
$= -\frac{9}{4} + \frac{9}{2} + \sqrt{3} - \sqrt{3}$

$=$ **$\frac{9}{4}$ units^2**



Check final answer with G. C. (time permitting)

(iv)



Since $x = -1$ and $x = 1$ are stationary points,

When $x = -1$, $y_{max} = (-1)^3 - 3(-1) + 1 = 3$

When $x = 1$, $y_{min} = (1)^3 - 3(1) + 1 = -1$

For $x^3 - 3x + 1 = k$ to have three real distinct roots,

$y_{min} < k < y_{max}$

∴ Set of values of $k = \{k \in \mathbb{R}: -1 < k < 3\}$

Note:

- Final answer expressed in set notation as question asks for <u>set</u> of values of k.
- Three real distinct roots ⇒ no equal sign

7. **Topic: Differential Equations**

Given $\frac{d\theta}{dt} \propto (20-\theta)$

$$\Rightarrow \frac{d\theta}{dt} = k(20-\theta), \text{ where } k \text{ is a constant} \ldots\ldots\ldots\ldots (1)$$

When $t = 0$, sub $\theta = 10$ and $\frac{d\theta}{dt} = 1$ into (1)

$$1 = k(20-10)$$
$$k = \frac{1}{20-10} = \frac{1}{10}$$
$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{10}(20-\theta)$$
$$\int \frac{1}{20-\theta}\,d\theta = \int \frac{1}{10}\,dt$$

Variable separable form
$\int g(y)\,dy = \int f(x)\,dx$

$$-\ln|20-\theta| = \frac{1}{10}t + c$$

$\int \frac{1}{x}\,dx = \ln|x| + c$

When $t = 0, \theta = 10$

$$-\ln|20-10| = 0 + c$$
$$c = -\ln 10$$
$$\Rightarrow -\ln|20-\theta| = \frac{1}{10}t - \ln 10$$
$$\ln(20-\theta) = -\frac{1}{10} + \ln 10$$

θ is always ≤ 20

$$20-\theta = e^{-\frac{1}{10}t} \cdot e^{\ln 10}$$

$e^{\ln x} = x$

$$20-\theta = 10e^{-\frac{1}{10}t}$$
$$\therefore \theta = \mathbf{20 - 10e^{-\frac{1}{10}t}} \textbf{ (shown)} \ldots\ldots\ldots\ldots (2)$$

Sub $\theta = 15$ into (2),

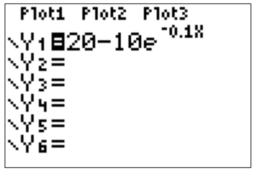
$$15 = 20 - 10e^{-\frac{1}{10}t}$$
$$10e^{-\frac{1}{10}t} = 5$$
$$e^{-\frac{1}{10}t} = \frac{1}{2}$$
$$t = -10\ln\frac{1}{2}$$
$$= \mathbf{10 \ln 2 \text{ mins}}$$

From (2), $\theta = 20 - 10e^{-\frac{1}{10}t}$

As $t \to \infty$, $e^{-\frac{1}{10}t} \to 0 \Rightarrow \theta \to 20$

$\therefore \theta$ approaches 20°C for large values of t.

Using G.C. (refer to Appendix for detailed steps),



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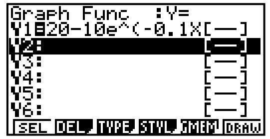

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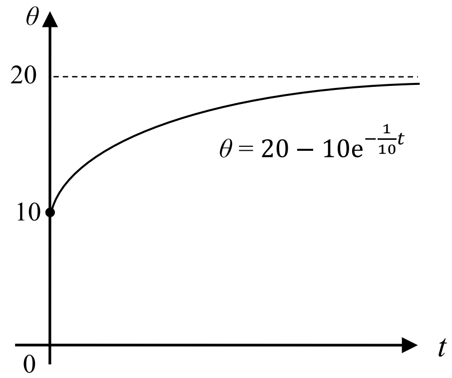
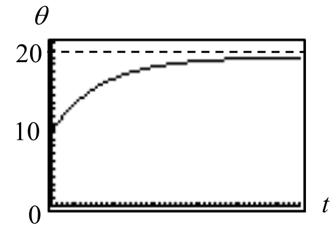




8. **Topic: Complex Numbers (Polar Form, Loci & Argand Diagrams)**

(i)

$$z_1 = 1 + \mathrm{i}\sqrt{3}$$

$$|z_1| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta_1 = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$\therefore z_1 = \mathbf{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$$

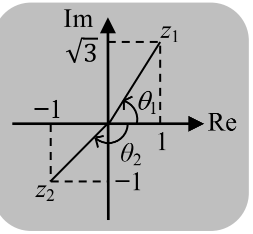
$$z_2 = -1 - \mathrm{i}$$

$$|z_2| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta_2 = -\pi + \tan^{-1}\left(\frac{-1}{-1}\right) = -\frac{3\pi}{4}$$

$$\therefore z_2 = \mathbf{\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\ \sin\left(-\frac{3\pi}{4}\right)\right]}$$

Polar Form:
$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)$
where $r = |z| = \sqrt{x^2 + y^2}$
$\theta = \tan^{-1}\frac{y}{x}$



(ii)

$$z_1 = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right) = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$$

$$z_2 = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\ \sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}$$

$$\frac{z_1}{z_2} = \frac{2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}}{\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}}$$

$$= \frac{2}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\left[\frac{\pi}{3} - \left(-\frac{3\pi}{4}\right)\right]}$$

$$= \sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{13}{12}\pi\right)}$$

$$= \sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{11}{12}\pi\right)}$$

$$= \sqrt{2}\left[\cos\left(-\frac{11}{12}\pi\right) + \mathrm{i}\sin\left(-\frac{11}{12}\pi\right)\right]$$

$$\therefore \left(\frac{z_1}{z_2}\right)^* = \sqrt{2}\left[\cos\left(-\frac{11}{12}\pi\right) - \mathrm{i}\sin\left(-\frac{11}{12}\pi\right)\right]$$

$$= \mathbf{\sqrt{2}\left[\cos\left(\frac{11}{12}\pi\right) + i\sin\left(\frac{11}{12}\pi\right)\right]}$$

Exponential Form:
$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}$

Principal argument ($-\pi < \theta \le \pi$)
$\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \frac{13\pi}{12} - 2\pi = -\frac{11\pi}{12}$

Complex conjugate
$z = a + \mathrm{i}b$
$z^* = a - \mathrm{i}b$

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

ALTERNATIVE APPROACH

$$\left|\left(\frac{z_1}{z_2}\right)^*\right| = \left|\frac{z_1}{z_2}\right|$$

$$= \frac{|z_1|}{|z_2|}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$|z^*| = |z|$

$$\arg\left(\frac{z_1}{z_2}\right)^* = -\arg\left(\frac{z_1}{z_2}\right)$$

$$= -\left[\frac{\pi}{3} - \left(-\frac{3\pi}{4}\right)\right]$$

$$= -\frac{13\pi}{12}$$

$$= \frac{11\pi}{12}$$

$\arg(z^*) = -\arg(z)$

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Principal argument ($-\pi < \theta \le \pi$)
$\mathrm{Arg}\left(\frac{z_1}{z_2}\right)^* = -\frac{13\pi}{12} + 2\pi = \frac{11\pi}{12}$

$$\therefore \left(\frac{z_1}{z_2}\right)^* = \mathbf{\sqrt{2}\left[\cos\left(\frac{11}{12}\pi\right) + i\sin\left(\frac{11}{12}\pi\right)\right]}$$

(iii) (a)

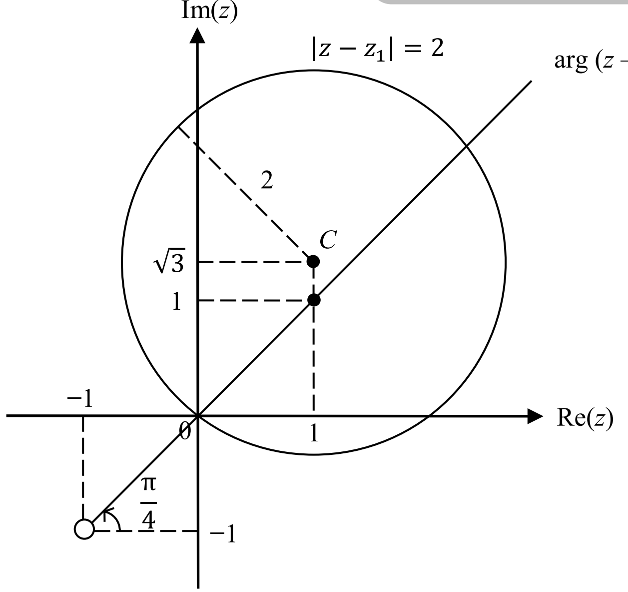
$$|z - z_1| = 2$$
$$|z - (1 + \mathrm{i}\sqrt{3})| = 2$$

⇒ circle with center $C\,(1, \sqrt{3})$ and radius 2 units.

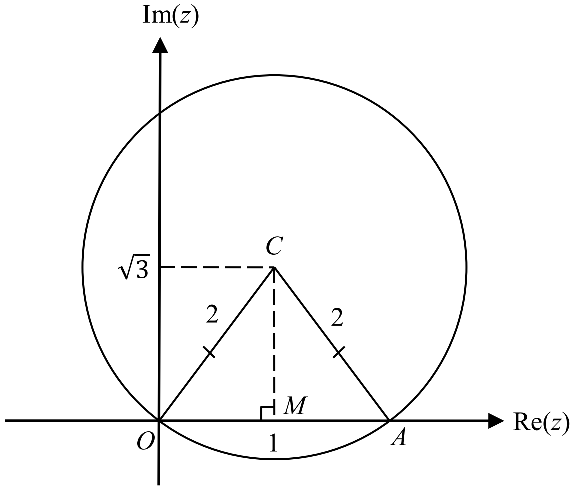
(b)

$$\arg(z - z_2) = \frac{\pi}{4}$$
$$\arg[z - (-1 - \mathrm{i})] = \frac{\pi}{4}$$

⇒ straight line beginning at $(-1, -1)$ with an angle of $\frac{\pi}{4}$ rad with respect to the Re(z) axis.



(iv) $CO = CA = 2 \Rightarrow \Delta\, COA$ is isosceles.
$CM \perp OA \Rightarrow OM = MA = 1$.
$\therefore OA = OM + MA = 1 + 1 = 2$.



∴ The locus $|z - z_1| = 2$ meets the positive real axis at (2, 0).

ALTERNATIVE APPROACH

Cartesian equation of $|z - z_1| = 2$ loci:

$$(x - 1)^2 + (y - \sqrt{3})^2 = 2^2$$

Sub $y = 0$,

$$(x - 1)^2 + 3 = 2^2$$
$$(x - 1)^2 = 1$$
$$x - 1 = \pm 1$$
$$x = 0 \text{ (reject)}, 2$$

∴ The locus $|z - z_1| = 2$ meets the positive real axis at (2, 0).

9. **Topic: Differentiation (Maxima & Minima)**

(i)

$$\text{Volume of box} = x(3x)y$$
$$300 = 3x^2y$$
$$y = \frac{100}{x^2} \text{ cm} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

External surface area of box, $A_1 = x(3x) + 2yx + 2y(3x)$
$= (3x^2 + 8xy)$ cm^2

External surface area of lid, $A_2 = x(3x) + 2(ky)x + 2(ky)(3x)$
$= (3x^2 + 8kxy)$ cm^2

Total external surface area of the box and the lid,

$$\begin{aligned} A &= A_1 + A_2 \\ &= (3x^2 + 8xy) + (3x^2 + 8kxy) \\ &= 6x^2 + 8(1+k)xy \\ &= 6x^2 + 8(1+k)x\left(\frac{100}{x^2}\right) \end{aligned}$$

Sub $y = \frac{100}{x^2}$

$$A = 6x^2 + \frac{800(1+k)}{x}$$
$$\frac{dA}{dx} = 12x - \frac{800(1+k)}{x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$\frac{d^2A}{dx^2} = 12 + \frac{1600(1+k)}{x^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Stationary point of A occurs when $\frac{dA}{dx} = 0$.

From (2), $12x - \frac{800(1+k)}{x^2} = 0$

$$12x = \frac{800(1+k)}{x^2}$$
$$x^3 = \frac{200(1+k)}{3}$$
$$x = \left[\frac{200(1+k)}{3}\right]^{\frac{1}{3}}$$

Using second derivative test, sub $x = \left[\frac{200(1+k)}{3}\right]^{\frac{1}{3}}$ into (3),

$$\begin{aligned} \frac{d^2A}{dx^2} &= 12 + \frac{1600(1+k)}{\left(\left[\frac{200(1+k)}{3}\right]^{\frac{1}{3}}\right)^3} \\ &= 12 + 1600(1+k) \times \frac{3}{200(1+k)} \\ &= 36 > 0 \end{aligned}$$

Second Derivative Test:

Sign of $\frac{d^2y}{dx^2}$	Nature of stationary point
–	Maximum
+	Minimum
0	Point of inflexion

$\Rightarrow$ **A is minimum when $x = \left[\frac{200(1+k)}{3}\right]^{\frac{1}{3}}$**

(ii) From (1)

$$\begin{aligned} y &= \frac{100}{x^2} \\ \frac{y}{x} &= \frac{100}{x^3} \\ &= \frac{100}{\left(\left[\frac{200(1+k)}{3}\right]^{\frac{1}{3}}\right)^3} \\ &= 100\left[\frac{3}{200(1+k)}\right] \\ &= \mathbf{\frac{3}{2(1+k)}} \end{aligned}$$

Sub $x = \left[\frac{200(1+k)}{3}\right]^{\frac{1}{3}}$

(iii) Given

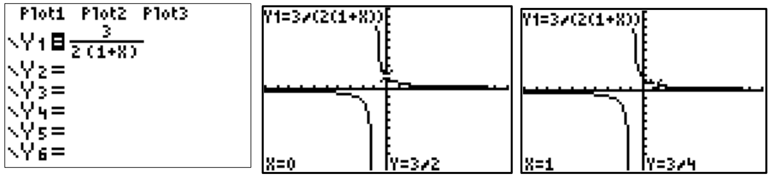
$$\begin{aligned} 0 &< k \le 1 \\ 1 &< 1+k \le 2 \\ 2 &< 2(1+k) \le 4 \\ \frac{1}{4} &\le \frac{1}{2(1+k)} < \frac{1}{2} \\ \frac{3}{4} &\le \frac{3}{2(1+k)} < \frac{3}{2} \\ \therefore \mathbf{\frac{3}{4}} &\le \mathbf{\frac{y}{x}} < \mathbf{\frac{3}{2}} \end{aligned}$$

Manipulate k to get $\frac{3}{2(1+k)} = \frac{y}{x}$

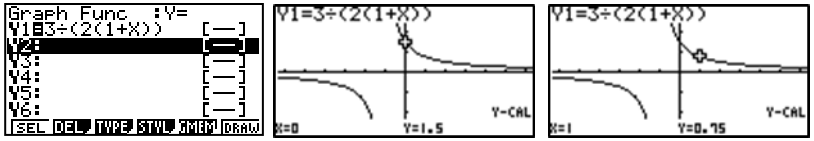
ALTERNATIVE APPROACH

$\frac{y}{x} = \frac{3}{2(1+k)} \rightarrow$ Let $Y = \frac{y}{x}$ and $X = k \Rightarrow Y = \frac{3}{2(1+X)}$

Using G.C. to obtain range of Y for $0 < X \leq 1$ (refer to Appendix for detailed steps),



TI-84 Plus



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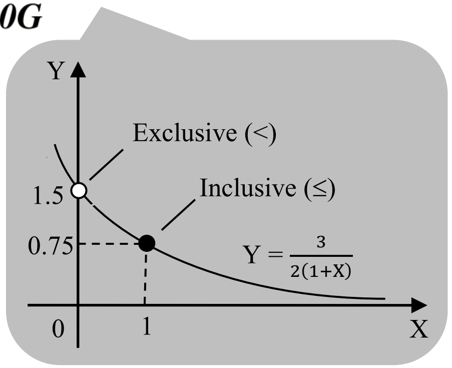


Given $0 < k \leq 1$

$\therefore \ \frac{3}{4} \leq \frac{y}{x} < \frac{3}{2}$

(iv) For the box to have square ends, $y = x$.

From (ii),

$$\frac{y}{x} = \frac{3}{2(1+k)}$$

$$1 = \frac{3}{2(1+k)}$$

$$1 + k = \frac{3}{2}$$

$$\boldsymbol{k} = \frac{\mathbf{1}}{\mathbf{2}}$$

10.

Topic: Vectors in Three Dimensions (Lines &Planes)

(i) l: $\frac{x-10}{-3} = \frac{y+1}{6} = \frac{z+3}{9}$

$$\Rightarrow \ \mathbf{r} = \begin{pmatrix} 10 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 6 \\ 9 \end{pmatrix}, \lambda \in \mathbb{R} \ldots\ldots\ldots\ldots\ldots (1)$$

$$\mathbf{r} = \begin{pmatrix} 10 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \text{ where } \mu = 3\lambda$$

$$\Rightarrow \text{ direction vector of } l, \ \mathbf{d} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

p: $x - 2y - 3z = 0$

$$\Rightarrow \ \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\Rightarrow \text{ normal of plane } p, \ \mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = -\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Since $\mathbf{n} = -\mathbf{d}$, l is parallel to normal, $\mathbf{n}$.

∴*l* is perpendicular to plane *p*. (shown)

Equation of Line:

Cartesian form

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

Vector form

$$\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \lambda \in \mathbb{R}$$

Equation of Plane:

Cartesian form

$$n_1 x + n_2 y + n_3 z = D$$

Standard form

$$\mathbf{r} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = D$$

(ii) From (1),

$$\mathbf{r} = \begin{pmatrix} 10 - \mu \\ -1 + 2\mu \\ -3 + 3\mu \end{pmatrix} \ldots\ldots\ldots (3)$$

At intersection point, sub (3) into (2),

$$\begin{pmatrix} 10 - \mu \\ -1 + 2\mu \\ -3 + 3\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0$$

$$10 - \mu + 2 - 4\mu + 9 - 9\mu = 0$$

$$21 - 14\mu = 0$$

$$\mu = \frac{3}{2}$$

Sub $\mu = \frac{3}{2}$ into (2),

$$\mathbf{r} = \begin{pmatrix} 10 - \frac{3}{2} \\ -1 + 2\left(\frac{3}{2}\right) \\ -3 + 3\left(\frac{3}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{17}{2} \\ 2 \\ \frac{3}{2} \end{pmatrix}$$

∴ Coordinates of the point of intersection of l and p is $\left(\frac{17}{2}, 2, \frac{3}{2}\right)$.

(iii) Sub A (−2, 23, 33) into (1),

$$\begin{pmatrix} -2 \\ 23 \\ 33 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

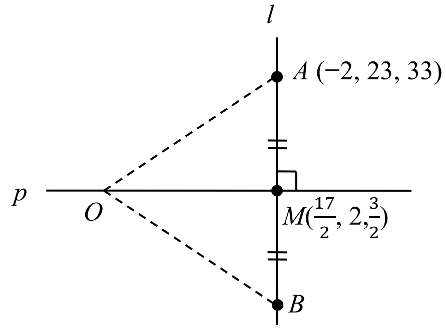
$$\mu \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -12 \\ 24 \\ 36 \end{pmatrix}$$

$$\mu \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 12 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$\mu = 12$$

Since there is a valid solution for μ, point A lies on l.

From (2), $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0 \Rightarrow$ Origin O lies on plane p.



Using Midpoint Theorem,

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$

$$\overrightarrow{OB} = 2\overrightarrow{OM} - \overrightarrow{OA}$$

$$= 2\begin{pmatrix} \frac{17}{2} \\ 2 \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} -2 \\ 23 \\ 33 \end{pmatrix}$$

$$= \begin{pmatrix} 19 \\ -19 \\ -30 \end{pmatrix}$$

Midpoint Theorem:
If M is the midpoint of AB, then $\mathbf{m} = \frac{\mathbf{a} + \mathbf{b}}{2}$

∴ Coordinates of the point B is (19, −19, − 30).

(iv) Area of $\Delta OAB = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

$$= \frac{1}{2}\left|\begin{pmatrix}-2\\23\\33\end{pmatrix} \times \begin{pmatrix}19\\-19\\-30\end{pmatrix}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix}-63\\567\\-399\end{pmatrix}\right|$$

$$= \frac{1}{2}\sqrt{(-63)^2 + (567)^2 + (-399)^2}$$

$$= 348.087$$

$$\approx \mathbf{348\ unit^2}$$

Vector Product:

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}$$

11. **Topic: Differentiation, Parametric Equations & Graphs**

(i) Given $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$

$$\frac{dx}{dt} = 1 - \frac{1}{t^2} = \frac{t^2-1}{t^2} \qquad \frac{dy}{dt} = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^2+1}{t^2-1}$$

Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

At point P, $t = p$,

$$x = p + \frac{1}{p} \quad \text{and} \quad y = p - \frac{1}{p}$$

$$\frac{dy}{dx} = \frac{p^2+1}{p^2-1}$$

Gradient of tangent at point P.

Gradient of tangent at point $(x_1, y_1) = \frac{y-y_1}{x-x_1}$

Equation of tangent at point P:

$$y - \left(p - \frac{1}{p}\right) = \frac{p^2+1}{p^2-1}\left[x - \left(p + \frac{1}{p}\right)\right]$$

$$y - \left(\frac{p^2-1}{p}\right) = \frac{p^2+1}{p^2-1}(x) - \left(\frac{p^2+1}{p^2-1}\right)\left(\frac{p^2+1}{p}\right)$$

$$p(p^2-1)\,y - (p^2-1)(p^2-1) = p\,(p^2+1)\,x - (p^2+1)(p^2+1)$$

$$p(p^2-1)\,y - p^4 + 2p^2 - 1 = p\,(p^2+1)\,x - p^4 - 2p^2 - 1$$

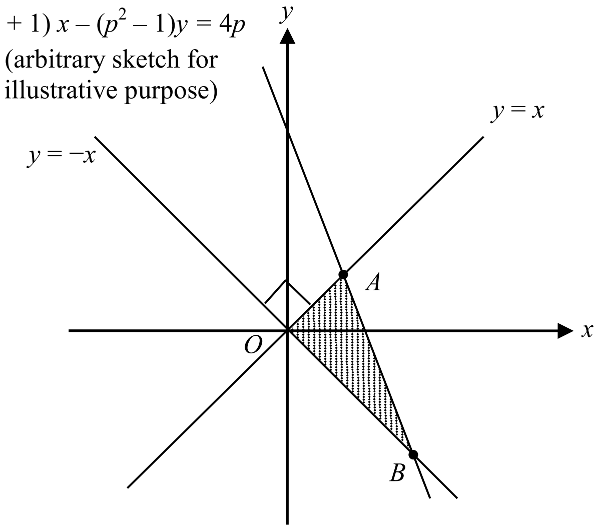
$$p(p^2-1)\,y = p\,(p^2+1)\,x - 4p^2$$

$$(p^2-1)y = (p^2+1)\,x - 4p$$

$$\mathbf{(p^2+1)\,x - (p^2-1)y = 4p \text{ (shown)}}$$

(ii)

$(p^2+1)x-(p^2-1)y=4p$
(arbitrary sketch for illustrative purpose)



Equation of tangent at P,

$(p^2+1)x-(p^2-1)y = 4p$ ……………………(1)

At point A, sub $y=x$ into (1)

$$[(p^2+1)-(p^2-1)]x = 4p$$
$$x = 2p \Rightarrow y = 2p$$
$$\Rightarrow A\,(2p, 2p)$$

At point B, sub $y=-x$ into (1)

$$[p^2+1+p^2-1]x = 4p$$
$$2p^2x = 4p$$
$$x = \frac{2}{p} \Rightarrow y = -\frac{2}{p}$$
$$\Rightarrow B\left(\frac{2}{p}, -\frac{2}{p}\right)$$

$OA \perp OB$

$$\text{Area of } \Delta OAB = \frac{1}{2}|OA||OB|$$
$$= \frac{1}{2}\left(\sqrt{(2p)^2+(2p)^2}\right)\left(\sqrt{\left(\frac{2}{p}\right)^2+\left(\frac{-2}{p}\right)^2}\right)$$
$$= \frac{1}{2}\left(p\sqrt{8}\right)\left(\frac{1}{p}\sqrt{8}\right)$$
$$= \frac{1}{2}(8)$$
$$= 4 \text{ units}^2$$

∴ Area of ΔOAB is independent of p. (shown)

(iii)

$x = t+\frac{1}{t}$ …………………………(2)

$y = t-\frac{1}{t}$ …………………………(3)

(2) + (3)

$$x+y = 2t$$
$t = \frac{x+y}{2}$ …………………………(4)

Sub (4) into (2)

$$x = \frac{x+y}{2}+\frac{2}{x+y}$$

Eliminate t, so that only x and y are left in the equation.

$$2x(x+y) = (x+y)^2+4$$
$$2x^2+2xy = x^2+2xy+y^2+4$$
$$4+y^2 = x^2$$
$$x^2-y^2 = 4$$
$$\boldsymbol{\frac{x^2}{2^2}-\frac{y^2}{2^2} = 1}$$

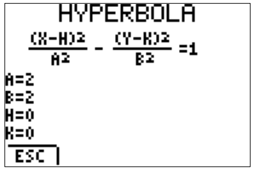
<u>Graph of Hyperbola</u>:

Equation: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

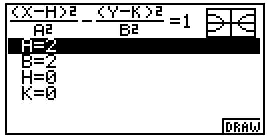
Center: (h, k)

Oblique Asymptotes: $y = k \pm \frac{b}{a}(x-h)$

Using G.C. (refer to Appendix for detailed steps),



TI-84 Plus



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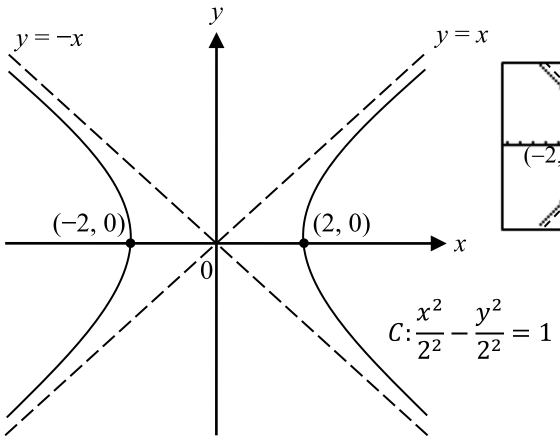
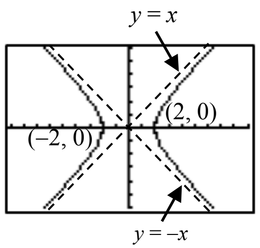




Asymptotes: $y = x$ and $y = -x$

x-intercepts: $(-2, 0)$ and $(2, 0)$

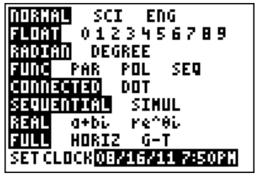
Appendix: Detailed G. C. Steps (for those still trapped in G. C. limbo)

Q5 (ii): Sketching the Inverse of a Function

TI-84 Plus

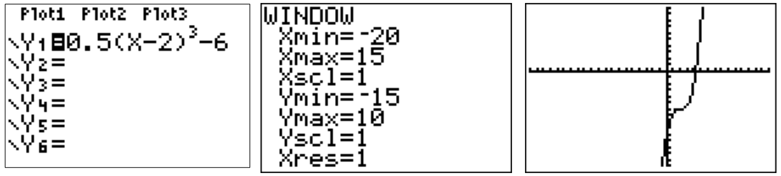
[MODE]

→ Ensure G. C. is in **FUNC** mode.



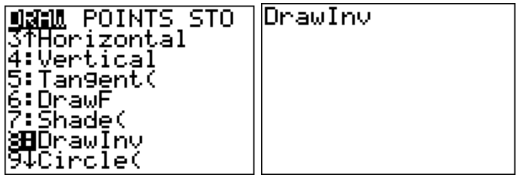
[Y=] [WINDOW] [GRAPH]

→ Plot f(x)



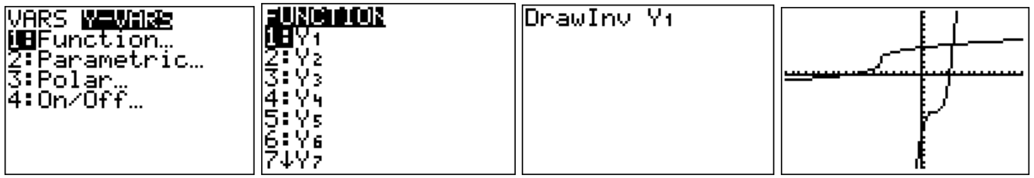
[2nd] [PRGM] [8]

→ Plot $f^{-1}(x)$



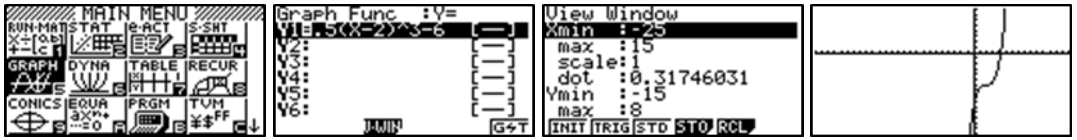
[VARS] [▶]

[ENTER] [ENTER] [ENTER]



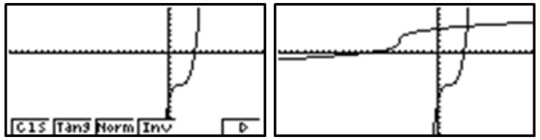
fx-9860G

→ Plot f(x)



[SHIFT] ↑ [EXIT] [F6]

→ Plot $f^{-1}(x)$



[F4]

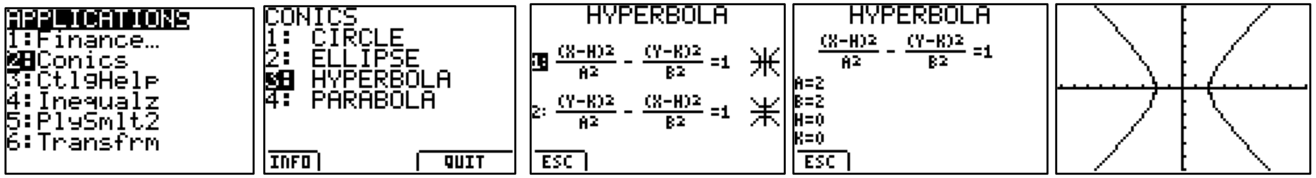
Q11: Sketching A Hyperbola

TI-84 Plus

APPS

→ Enter **Conics**.
→ Enter **HYPERBOLA**.

GRAPH



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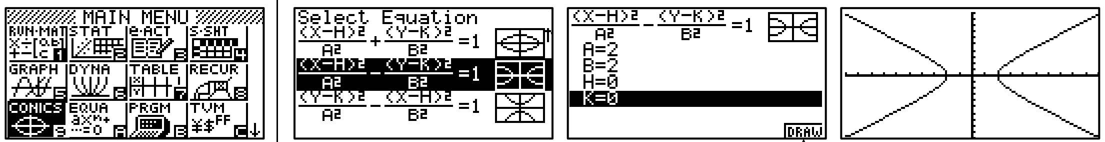
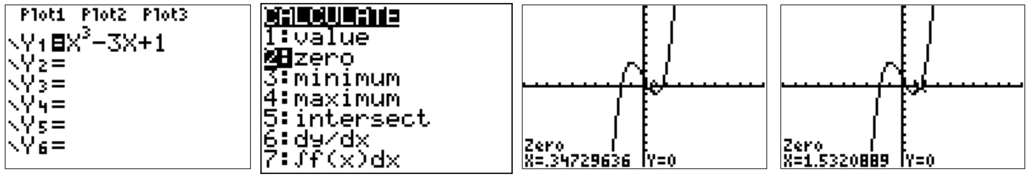


Q6 (i): Finding the Roots of a Function

TI-84 Plus

Method I: Using Zero Values of Graphs

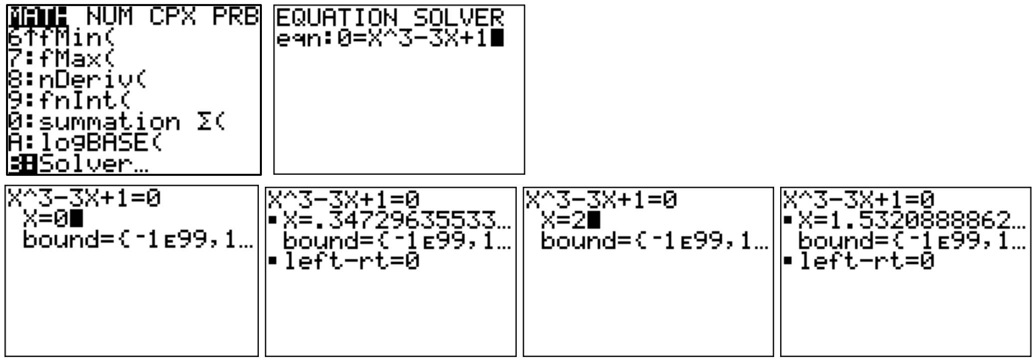
→ Define approximate bounds of x value to left/right of β.
→ Repeat steps for root γ.



Method II: Using Solver

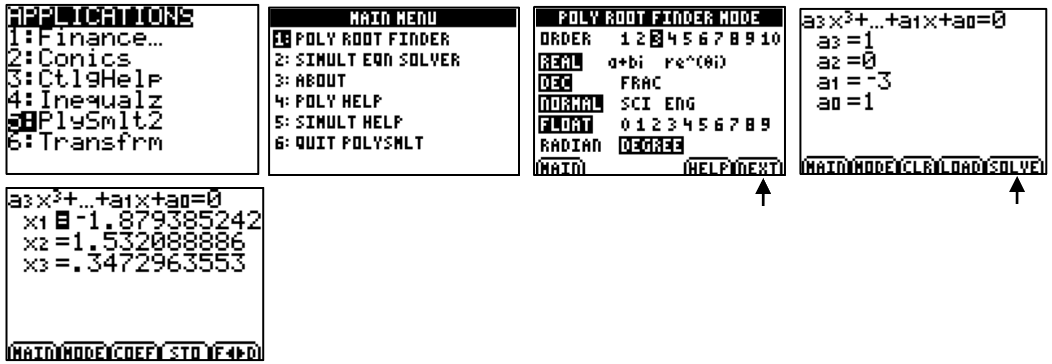
→Enter **Solver** (last item on MATH menu)
→ Enter equation.

→ Enter suitable approximate value of x (i.e. $x = 0$) for root β.
→ Enter suitable approximate value of x (i.e. $x = 2$) for root γ.



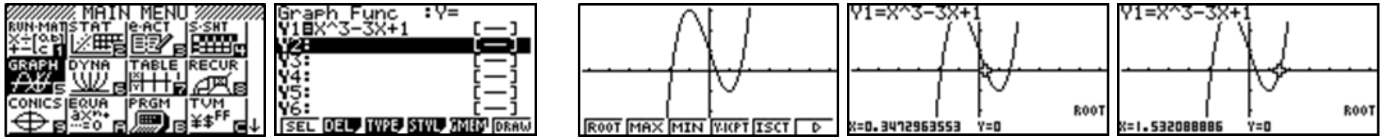
Method III: Using Poly Root Finder Application

→ Enter **PlySmlt2**



fx-9860G

Method I:
Find Roots in Graph



[F5]

Method II:
Using Polynomials in Equation

Q6 (ii): Finding Region Bounded by Curve

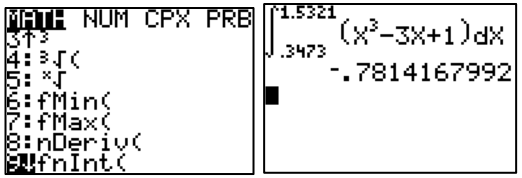
TI-84 Plus

Method I: Using "fnInt(" Function

MATH 9

→ Key in lower/upper limits, expression and the integrating variable (x).

ENTER



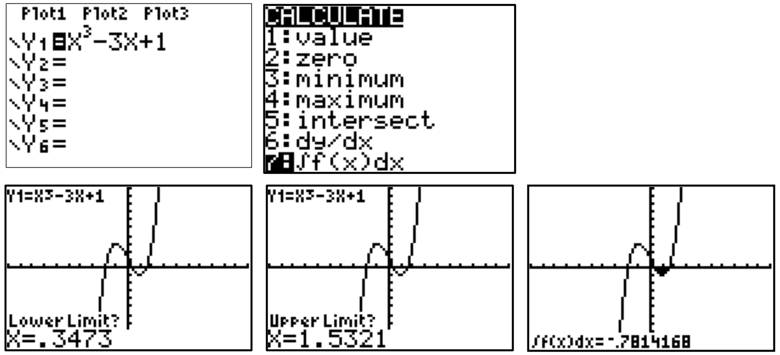
Method II: Using Integration Function in Graph

Y=

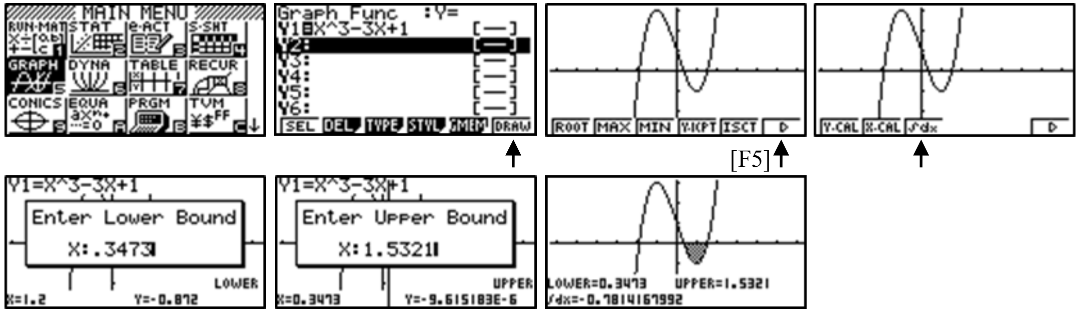
2nd TRACE 7

→ Enter β and γ as the values for lower and upper limit respectively.

ENTER

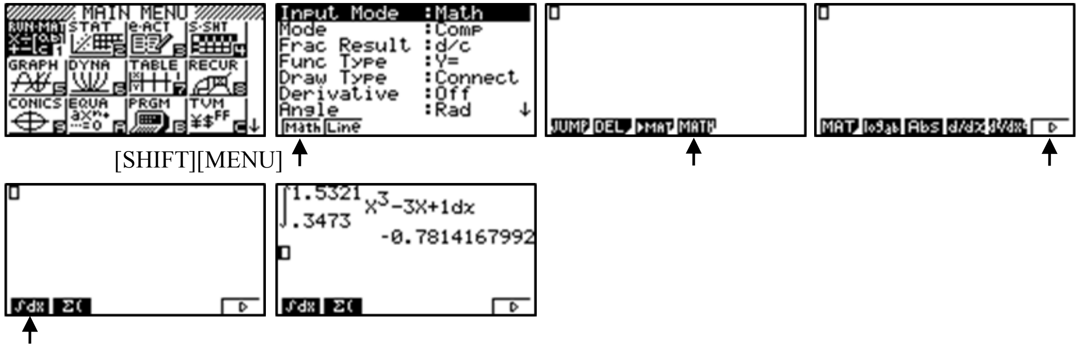


fx-9860G **Method I:Using Integration Function in Graph**



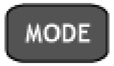
Method II: Using Integration Function in MATHInput Mode

→ Key in lower/upper limits and the expression.

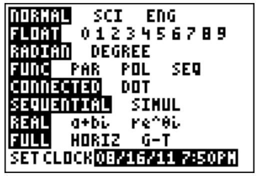


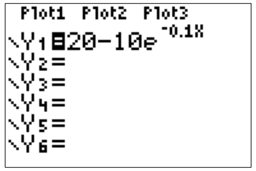
Q7: Graph Sketching

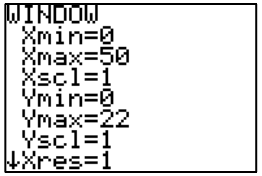
TI-84 Plus

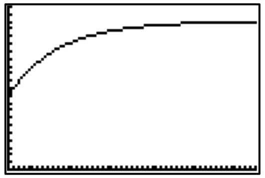


→ Ensure G. C. is in FUNC mode.









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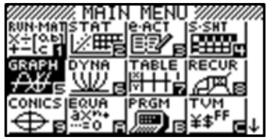
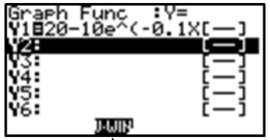
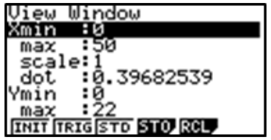
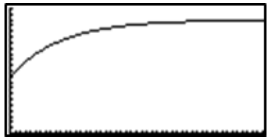




[SHIFT] ↑

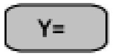




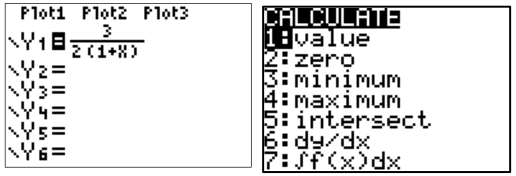
[EXIT] [F6]

Q9 (iii): Finding the Range of Values in a Function

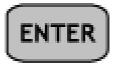
TI-84 Plus

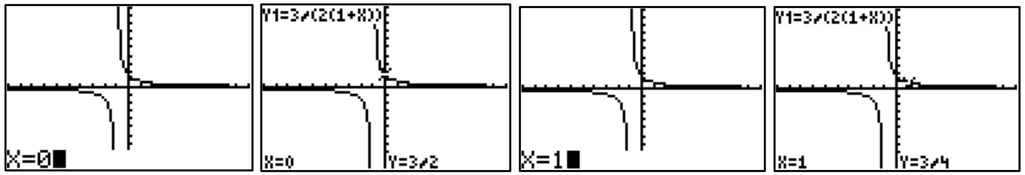


→ Enter expression

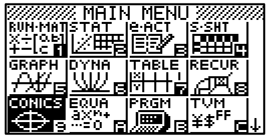


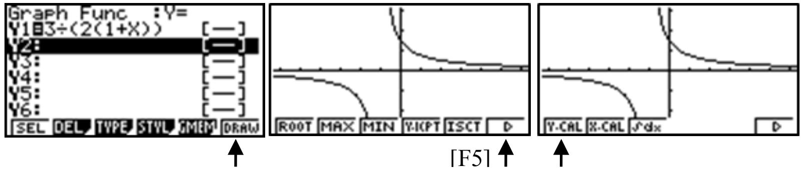
→ Enter X = 0 and X = 1to obtain the corresponding values of Y.





fx-9860G





[F5]

→ Enter X = 0 and X = 1to obtain the corresponding values of Y.