ADDITIONAL MATHEMATICS 4038/01

Paper 1 Suggested Solutions October/November 2010

1. **Topic: Polynomials**

(i) $f(x) = x^4 - x^3 + kx - 4$

Since $(x - 2)$ is a factor of f(x),

$\Rightarrow f(2) = 2^4 - 2^3 + k(2) - 4 = 0$

$16 - 8 + 2k - 4 = 0$

$k = \mathbf{-2}$

> Factor Theorem:
> $f(a) = 0 \Leftrightarrow (x - a)$ is a factor of f(x)

(ii) $f(x) = x^4 - x^3 - 2x - 4$

$f(-2) = (-2)^4 - (-2)^3 - 2(-2) - 4$

$= 16 + 8 + 4 - 4$

$= 24$

$\therefore$ Remainder when f(x) is divided by $(x + 2)$ = **24**.

> Remainder Theorem:
> f(x) divided by $(x - a)$
> $\Rightarrow$remainder is f(a)

2. **Topic: Further Trigonometric Identities, Integration**

(i) To show that $(\sin x + \cos x)^2 = 1 + \sin 2x$,

L.H.S. $= (\sin x + \cos x)^2$

$= \sin^2 x + 2\sin x\cos x + \cos^2 x$

$= \sin^2 x + \cos^2 x + 2\sin x \cos x$

$= 1 + \sin 2x$

= R.H.S.**(Shown)**

> Given in formula sheet:
> $\cos^2 A + \sin^2 A = 1$

> Double Angle Formula (given in formula sheet):
> $\sin 2A = 2\sin A\cos A$

(ii) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx = \int_0^{\frac{\pi}{2}} (1 + \sin 2x) \, dx$

$= \left[x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \left[\frac{\pi}{2} - \frac{1}{2}\cos 2\left(\frac{\pi}{2}\right)\right] - \left[0 - \frac{1}{2}\cos 2(0)\right]$

$= \frac{\pi}{2} - \frac{1}{2}\cos \pi + \frac{1}{2}$

$= \frac{\pi}{2} - \frac{1}{2}(-1) + \frac{1}{2}$

$= \mathbf{1 + \frac{\pi}{2}}$

> using proof from (i)

> $\int \sin(ax + b) \, dx = -\frac{1}{a}\cos(ax + b) + c$

3. **Topic: Quadratic Functions and Inequalities**

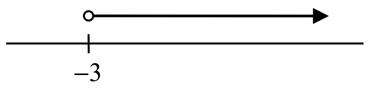
(i) $3(2 - x) < x + 18$

$6 - 3x < x + 18$

$6 - 18 < x + 3x$

$4x > -12$

$x > -3$



Solution set = $\{x : x > -3, x \in \mathbb{R}\}$

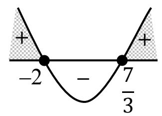
(ii) $3(x^2 - 5) > x - 1$

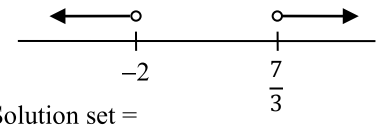
$3x^2 - 15 > x - 1$

$3x^2 - x - 14 > 0$

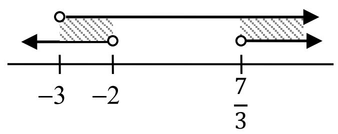
$(3x - 7)(x + 2) > 0$

$x < -2$ or $x > \frac{7}{3}$





Solution set = $\{x : x < -2, x \in \mathbb{R}\} \cup \left\{x : x > \frac{7}{3}, x \in \mathbb{R}\right\}$



> Note: Question asks for set NOT range of values of x. Hence final answer must be expressed in set notation.

$\therefore$ **The set of values of x which satisfy both inequalities**

$= \{x : -3 < x < -2, x \in \mathbb{R}\} \cup \left\{x : x > \frac{7}{3}, x \in \mathbb{R}\right\}$

4. **Topic: Applications of Differentiation (Rates of Change)**

(i) $y = \sin 2x - 3\cos x$

$\frac{dy}{dx} = 2\cos 2x + 3\sin x$

> $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$
> $\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$

When $x = \frac{\pi}{6}$,

$\frac{dy}{dx} = 2\cos 2\left(\frac{\pi}{6}\right) + 3\sin\left(\frac{\pi}{6}\right)$

$= (2)\left(\frac{1}{2}\right) + (3)\left(\frac{1}{2}\right)$

$= \mathbf{\frac{5}{2}}$

(ii) $\frac{dx}{dt} = 0.06$ units/s,

> Chain Rule: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

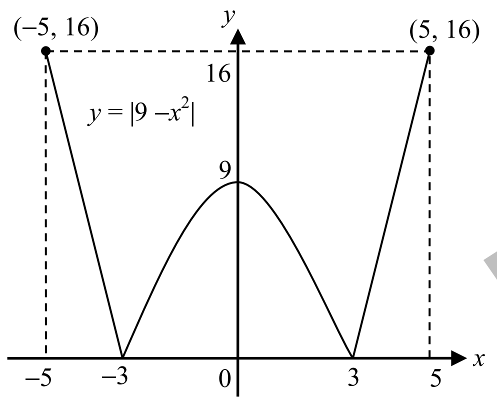
> Using value of $\frac{dy}{dx}$ obtained in part (i).

When $x = \frac{\pi}{6}$, $\frac{dy}{dt} = \frac{5}{2} \times 0.06$

$= \mathbf{0.15}$ **units/s**

5. **Topic: Modulus Functions**

(i) $y = |9 - x^2|$, $-5 \le x \le 5$



> y-intercept: when $x = 0$,
> $y = |9 - 0| = 9$
> x-intercept: when $y = 0$,
> $|9 - x^2| = 0$
> $x^2 = 9$
> $x = \pm 3$
> When $x = \pm 5$,
> $y = |9 - (\pm 5)^2|$
> $= |9 - 25|$
> $= |-16|$
> $= 16$

(ii) When $y = 27$, we have $27 = |9 - x^2|$.

$\Rightarrow$ $27 = 9 - x^2$ or $27 = -(9 - x^2)$

$x^2 = 9 - 27$ $27 = -9 + x^2$

$x^2 = -18$ (no solution) $x^2 = 36$

$x = \mathbf{\pm 6}$

> $f(x) = |g(x)|$
> $\Rightarrow f(x) = g(x)$ or $f(x) = -g(x)$

$\therefore$ ***x*-coordinates of the intersections are –6 and 6.**

6. **Topic: Differentiation and Integration**

> Product Rule: For $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

(i) $\frac{d}{dx}(xe^{2x}) = (1)e^{2x} + x(2e^{2x})$

$= \mathbf{e^{2x} + 2xe^{2x}}$

> $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$

(ii) From (i), we have

$\frac{d}{dx}(xe^{2x}) = e^{2x} + 2xe^{2x}$

Integrating both sides:

$$\int_0^1 \frac{d}{dx}(xe^{2x})\,dx = \int_0^1 e^{2x}\,dx + 2\int_0^1 xe^{2x}\,dx$$

$$[xe^{2x}]_0^1 = \left[\frac{1}{2}e^{2x}\right]_0^1 + 2\int_0^1 xe^{2x}\,dx$$

$$\left[(1)e^{2(1)} - (0)e^{2(0)}\right] = \left[\frac{1}{2}e^{2(1)} - \frac{1}{2}e^{2(0)}\right] + 2\int_0^1 xe^{2x}\,dx$$

$$2\int_0^1 xe^{2x}\,dx = e^2 - \frac{1}{2}e^2 + \frac{1}{2}$$

$$\int_0^1 xe^{2x}\,dx = \frac{1}{2}\left[\frac{1}{2}e^2 + \frac{1}{2}\right]$$

$$= \frac{e^2+1}{4}\ \textbf{(Shown)}$$

Alternative Method:

$$\int_0^1 xe^{2x}\,dx = \frac{1}{2}\int_0^1 2xe^{2x}\,dx$$
$$= \frac{1}{2}\left[\int_0^1 (2xe^{2x} + e^{2x} - e^{2x})\,dx\right]$$
$$= \frac{1}{2}\left[\int_0^1 (2xe^{2x} + e^{2x})\,dx - \int_0^1 e^{2x}\,dx\right]$$
$$= \frac{1}{2}\left[xe^{2x} - \frac{1}{2}e^{2x}\right]_0^1$$
$$= \frac{1}{2}\left[e^2 - \frac{e^2}{2}\right] - \frac{1}{2}\left[0 - \frac{e^0}{2}\right]$$
$$= \frac{1}{2}\left[\frac{e^2}{2}\right] + \frac{1}{4}$$
$$= \frac{e^2}{4} + \frac{1}{4}$$
$$= \frac{e^2+1}{4}\ \textbf{(Shown)}$$

From (i), $\int_0^1 e^{2x} + 2xe^x\,dx = [xe^{2x}]_0^1$

$\int_0^1 e^{ax}\,dx = \left[\frac{1}{a}e^{ax}\right]_0^1$

7. **Topic: Linear Law; Straight Line Graphs**

$$yx^n = k$$
$$\lg(yx^n) = \lg k$$
$$\lg y + n\lg x = \lg k$$
$$\lg y = -n\lg x + \lg k$$

$\lg(xy) = \lg x + \lg y$
$\lg x^n = n\lg x$

By denoting $Y = \lg y$ and $X = \lg x$, we plot the graph of $Y = -nX + \lg k$ using the following table of derived values for X and Y:

x	2	8	14	20
y	33.00	5.07	2.38	1.47
X (lg$x$)	0.301	0.903	1.146	1.301
Y (lg$y$)	1.519	0.705	0.377	0.167

From the graph,

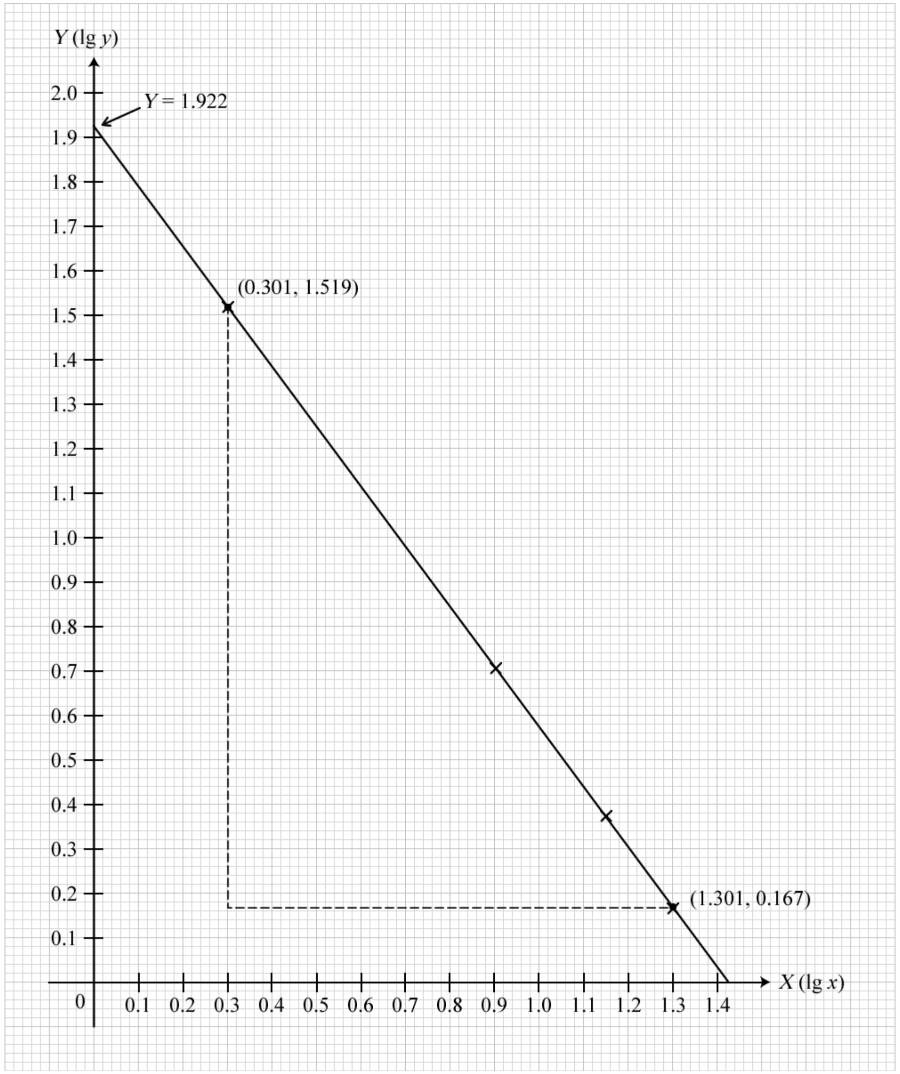
Gradient: $-n = \frac{1.519 - 0.167}{0.301 - 1.301} = -1.352$

$\therefore n = 1.352 \approx$ **1.35 (3 sig. fig.)**

Y-intercept: $\lg k = 1.922$

$\therefore k = 10^{1.922} = 83.56 \approx$ **83.6 (3 sig. fig.)**

$\lg y = x \Leftrightarrow y = 10^x$



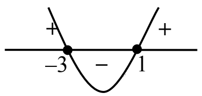
8. **Topic: Applications of Differentiation (Increasing and Decreasing Functions; Gradients, Tangent&Normals)**

$y = x^3 + 3x^2 - 9x + k \quad - (1)$

(i) $\frac{dy}{dx} = 3x^2 + 6x - 9$

When y is decreasing,

$$\frac{dy}{dx} < 0$$
$$3x^2 + 6x - 9 < 0$$
$$x^2 + 2x - 3 < 0$$
$$(x + 3)(x - 1) < 0$$
$$-3 < x < 1$$



> Note: Question asks for set NOT range of values of x. Hence final answer must be expressed in set notation.

∴ The set of values of $x = \{x : -3 < x < 1, x \in \mathbb{R}\}$

(ii) When the x-axis (i.e. $y = 0$) is tangent to the curve,

$$\frac{dy}{dx} = 0$$
$$3x^2 + 6x - 9 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x = -3 \quad \text{or} \quad x = 1$$

Hence, the points (–3, 0) or (1, 0) lie on the curve.

Sub $x = -3$ and $y = 0$ into (1),

$$0 = (-3)^3 + 3(-3)^2 - 9(-3) + k$$
$$0 = -27 + 27 + 27 + k$$
$$k = -27$$

Sub $x = 1$ and $y = 0$ into (1),

$$0 = (1)^3 + 3(1)^2 - 9(1) + k$$
$$0 = 1 + 3 - 9 + k$$
$$k = 5$$

∴ Possible values of $k = -27$ or 5

9. **Topic: Quadratic Equations (Sum & Product of Roots)**

For $3x^2 - 2x + 1 = 0$ with roots α and β,

$\Rightarrow$ Sum of roots: $\alpha + \beta = \frac{2}{3}$

Product of roots: $\alpha\beta = \frac{1}{3}$

> For $ax^2 + bx + c = 0$ with roots α and β,
> Sum of roots $\alpha + \beta = -\frac{b}{a}$
> Product of roots $\alpha\beta = \frac{c}{a}$

For the quadratic equation with roots $\alpha + 2\beta$ and $2\alpha + \beta$,

$$\begin{aligned}
\text{Sum of roots} &= (\alpha + 2\beta) + (2\alpha + \beta) \\
&= 3\alpha + 3\beta \\
&= 3(\alpha + \beta) \\
&= 3\left(\frac{2}{3}\right) \\
&= 2
\end{aligned}$$

> Sub $\alpha + \beta = \frac{2}{3}$

$$\begin{aligned}
\text{Product of roots} &= (\alpha + 2\beta)(2\alpha + \beta) \\
&= 2\alpha^2 + \alpha\beta + 4\alpha\beta + 2\beta^2 \\
&= 2\alpha^2 + 4\alpha\beta + 2\beta^2 + \alpha\beta \\
&= 2[\alpha^2 + 2\alpha\beta + \beta^2] + \alpha\beta \\
&= 2(\alpha + \beta)^2 + \alpha\beta \\
&= 2\left(\frac{2}{3}\right)^2 + \frac{1}{3} \\
&= \frac{11}{9}
\end{aligned}$$

> Sub $\alpha + \beta = \frac{2}{3}$, $\alpha\beta = \frac{1}{3}$

Hence quadratic equation with roots $\alpha + 2\beta$ and $2\alpha + \beta$:

$$x^2 - (\text{Sum of roots})\,x + (\text{Product of roots}) = 0$$
$$x^2 - (2)x + \frac{11}{9} = 0$$
$$\mathbf{9x^2 - 18x + 11 = 0}$$

10. **Topic: Further Trigonometric Identities**

(i) To show that $\tan 75° = 2 + \sqrt{3}$,

L.H.S. $= \tan 75°$

$= \tan(45° + 30°)$

$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

Addition formula (given in formula sheet): $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$

$\tan 45° = 1$
$\tan 30° = \frac{1}{\sqrt{3}}$

$= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

Rationalise the denominator: $\frac{1}{\sqrt{a}+\sqrt{b}} \times \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$

$= \frac{3 + 2\sqrt{3} + 1}{3 - 1}$

$= 2 + \sqrt{3} =$ R.H.S. **(Shown)**

(ii) To show that $\sec^2 75° = 4 \tan 75°$,

L.H.S. $= \sec^2 75°$

$= 1 + \tan^2 75°$

Given in formula sheet: $\sec^2 A = 1 + \tan^2 A$

$= 1 + (2 + \sqrt{3})^2$

Sub $\tan 75° = 2 + \sqrt{3}$ from (i)

$= 1 + 4 + 4\sqrt{3} + 3$

$= 8 + 4\sqrt{3}$

$= 4(2 + \sqrt{3})$

$= 4 \tan 75°$

Sub $\tan 75° = 2 + \sqrt{3}$ from (i)

= R.H.S. **(Shown)**

11. **Topic: Integration; Applications of Differentiation (Stationary points, Maxima & Minima)**

(i) $\frac{dy}{dx} = \frac{8}{x^2} - 2$

$y = \int \left(\frac{8}{x^2} - 2\right) dx$

$y = -\frac{8}{x} - 2x + c$

Sub $y = 5$, $x = 1$ at $(1, 5)$,

$5 = -\frac{8}{1} - 2(1) + c$

$c = 15$

$\therefore y = -\frac{8}{x} - 2x + 15$

(ii) At stationary points, $\frac{dy}{dx} = 0$

$\frac{8}{x^2} - 2 = 0$

$\frac{8}{x^2} = 2$

$x^2 = 4$

$x = \pm 2$

(iii) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{8}{x^2} - 2\right) = (-2)\frac{8}{x^3} = -\frac{16}{x^3}$

Stationary points of curve $y = f(x)$:
Max. point: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$
Min. point: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$

When $x = 2$,

$y = -\frac{8}{2} - 2(2) + 15 = 7$

$\frac{d^2y}{dx^2} = -\frac{16}{2^3} = -2 < 0$

$\therefore$ **(2, 7) is a maximum point since $\frac{d^2y}{dx^2}$ is negative.**

When $x = -2$,

$y = -\frac{8}{-2} - 2(-2) + 15 = 23$

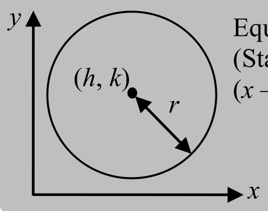
$\frac{d^2y}{dx^2} = -\frac{16}{(-2)^3} = 2 > 0$

$\therefore$ **(–2, 23) is a minimum point since $\frac{d^2y}{dx^2}$ is positive.**

12. **Topic: Coordinate Geometry (Circles)**

(i) Equation of circle with centre A (–3, 2) and radius 5:

$$(x+3)^2+(y-2)^2 = 5^2 \quad -(1)$$
$$x^2+6x+9+y^2-4y+4 = 25$$
$$\mathbf{x^2+y^2+6x-4y-12 = 0}$$



Equation of circle (Standard Form): $(x-h)^2+(y-k)^2=r^2$

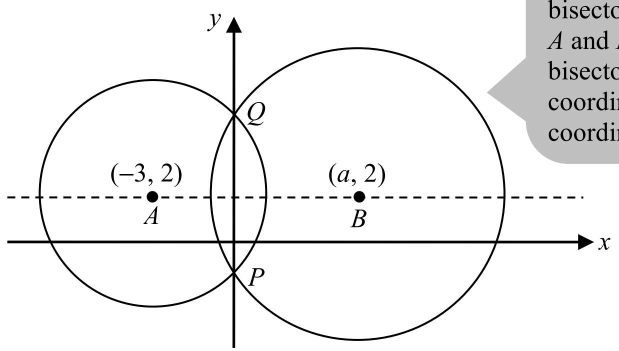
(ii) Since circle intersects the y-axis (i.e. $\underline{x=0}$), we sub $x = 0$ into (1),

$$(0+3)^2+(y-2)^2 = 25$$
$$(y-2)^2 = 16$$
$$y-2 = -4 \quad \text{or} \quad y-2 = 4$$
$$y = -2 \qquad\qquad y = 6$$

∴ The coordinates of P & Q are (0, –2) and (0, 6)

⇒ Length of PQ = 6 – (–2) = **8 units**

(iii) ***y*-coordinate of *B* = 2**



Since both circles share the vertical chord PQ, its ⊥ bisector passes through both A and B. And since this bisector is horizontal, the y-coordinate of B = y-coordinate of A.

(iv) Let the coordinates of B be (a, 2)

$$\text{Radius } QB = \sqrt{80}$$
$$\sqrt{(a-0)^2+(2-6)^2} = \sqrt{80}$$
$$a^2+16 = 80$$
$$a^2 = 64$$
$$a = 8 \text{ or } -8 \text{ (reject since } x\text{-coordinate of } B \text{ is positive)}$$

Length of Line Segment $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

∴ **The *x*-coordinate of *B* = 8.**

(v) Equation of circle with center B(8, 2) and radius = $\sqrt{80}$,

$$(x-8)^2+(y-2)^2 = \left(\sqrt{80}\right)^2$$
$$x^2-16x+64+y^2-4y+4 = 80$$
$$x^2+y^2-16x-4y-12 = 0$$

Equation of circle with centre $(-g, -f)$ and radius r (General Form): $x^2+y^2+2gx+2fy+c=0$

Comparing with $x^2+y^2-2gx-2fy-c=0$

⇒ $2g = -16$ ⇒ $\mathbf{g = -8}$

⇒ $2f = -4$ ⇒ $\mathbf{f = -2}$

⇒ $\mathbf{c = -12}$