ADDITIONAL MATHEMATICS 4038/02
Paper 2 Suggested Solutions October/November 2008

1. **Topic: Exponential Functions**

(i) Given: $V = 10000e^{-pt}$

When the man bought the motorcycle, $t = 0$.

$\therefore$ value of motorcycle when bought $= 10000e^0$
$= \mathbf{\$10000}$

(ii) When $t = 12$, $v = 4000$:

$4000 = 10000e^{-p(12)}$

$0.4 = e^{-12p}$

$\ln 0.4 = -12p \ln e$

$p = \frac{\ln 0.4}{-12}$

≈ 0.076358

$V = 10000e^{-0.076358t}$ ………. (1)

Sub $t = 18$ into (1): Value after 18 months $= 10000e^{-0.076358(18)}$
≈ 2529.8
$\approx$ **\$2530 (3 s.f.)**

(iii) Sub $v = 1000$ into (1):

$1000 = 10000e^{-0.076358t}$

$0.1 = e^{-0.076358t}$

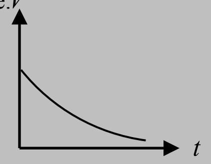
$\ln 0.1 = -0.076358t \ln e$

$t = \frac{\ln 0.1}{-0.076358}$

≈ 30.2 months

≈ 30 months (nearest month)

Note: $V <$ \$1000 when $t = 31$ months, since the e^{-pt} curve describes the depreciation of value over time.



$\therefore$ **age of motorcycle when expected value is \$1000 is 30 months.**

2. **Topic: Quadratic Equations (Sum & Product of Roots)**

$2x^2 - 4x + 3 = 0 \Rightarrow a = 2, b = -4, c = 3$

Sum of roots: $\alpha + \beta = -\frac{b}{a} = -\frac{-4}{2} = 2$

Product of roots: $\alpha\beta = \frac{c}{a} = \frac{3}{2}$

New sum of roots:

$(\alpha^2 + 2) + (\beta^2 + 2) = \alpha^2 + \beta^2 + 4$

$= [(\alpha + \beta)^2 - 2\alpha\beta] + 4$

$= (2)^2 - 2\left(\frac{3}{2}\right) + 4$ — Sub $\alpha + \beta = 2, \alpha\beta = \frac{3}{2}$

$= 5$

Useful expression: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

New product of roots:

$(\alpha^2 + 2)(\beta^2 + 2) = \alpha^2\beta^2 + 2\alpha^2 + 2\beta^2 + 4$

$= (\alpha\beta)^2 + 2(\alpha^2 + \beta^2) + 4$

$= (\alpha\beta)^2 + 2[(\alpha + \beta)^2 - 2\alpha\beta] + 4$

$= \frac{9}{4} + 2\left[4 - 2\left(\frac{3}{2}\right)\right] + 4$ — Sub $\alpha + \beta = 2, \alpha\beta = \frac{3}{2}$

$= \frac{9}{4} + 2 + 4$

$= 8\frac{1}{4} = \frac{33}{4}$

$\therefore$ equation with roots $\alpha^2 + 2, \beta^2 + 2$:

$x^2 - 5x + \frac{33}{4} = 0$ — x^2 – (Sum of roots)x + (Product of roots) = 0

$\Rightarrow \mathbf{4x^2 - 20x + 33 = 0}$

3. **Topic: Trigonometry (Trigonometric Identities & Equations)**

(i) $\tan A + \cot A = 2\operatorname{cosec} 2A$

L.H.S.: $\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}$

$= \frac{1}{\sin A \cos A}$

> $\sin^2 A + \cos^2 A = 1$

$= \frac{1}{\frac{2 \sin A \cos A}{2}}$

$= \frac{2}{\sin 2A}$

> $\sin 2A = 2 \sin A \cos A$

$= 2 \operatorname{cosec} 2A$

= R.H.S. **(Proved)**

(ii) $\tan A + \cot A = 3$

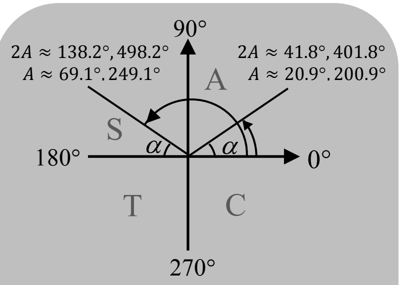
From (i): $2 \operatorname{cosec} 2A = 3$

$\frac{2}{\sin 2A} = 3$

$\sin 2A = \frac{2}{3}$

$\Rightarrow$ Basic angle $\alpha = \sin^{-1}\left(\frac{2}{3}\right)$

$\approx 41.81°$



> $\alpha \approx 41.81°$
> $0° \le A \le 360°$
> $0° \le 2A \le 720°$

$2A \approx 41.81°,\ 180° - 41.81°,$
$360° + 41.81°,\ 360° + 180° - 41.81°$
$\approx 41.81°,\ 138.19°,\ 401.81°,\ 498.19°$

$\therefore$ **A = 20.9°, 69.1°, 200.9°, 249.1°**

4. **Topic: Logarithms**

(i) $2 + \log_3(3x - 7) = \log_3(2x - 3)$

$\log_3(2x - 3) - \log_3(3x - 7) = 2$

$\log_3\left(\frac{2x-3}{3x-7}\right) = 2$

> Quotient Law:
> $\log_a m - \log_a n = \log_a \frac{m}{n}$

$\frac{2x-3}{3x-7} = 3^2$

> $y = a^x \Leftrightarrow x = \log_a y$

$2x - 3 = 9(3x - 7)$

$2x - 3 = 27x - 63$

$60 = 25x$

$\boldsymbol{x = 2.4}$

(ii) $3 \log_5 y - \log_y 5 = 2$

$3 \log_5 y - \frac{1}{\log_5 y} = 2$

> Change of Base of Log:
> $\log_a b = \frac{1}{\log_b a}$

Let $\log_5 y = x$.

$3x - \frac{1}{x} = 2$

$3x^2 - 1 = 2x$

$(3x + 1)(x - 1) = 0$

$x = -\frac{1}{3}$ or 1

$\Rightarrow \log_5 y = -\frac{1}{3}$ or 1

$y = 5^{-\frac{1}{3}}$ or 5^1

$\therefore$ **$y = \frac{1}{\sqrt[3]{5}}$ or 5**

5. **Topic: Polynomials (Factor Theorem & Remainder Theorem)**

(i) $f(x) = 2(x^2 - 3x + 1)(x + 1)(x - 2)$
$= (2x^2 - 6x + 2)(x^2 - x - 2)$
$= 2x^4 - 2x^3 - 4x^2 - 6x^3 + 6x^2 + 12x + 2x^2 - 2x - 4$
$= \mathbf{2x^4 - 8x^3 + 4x^2 + 10x - 4}$

> Factor Theorem:
> $f(a) = 0 \Leftrightarrow (x-a)$ is a factor of $f(x)$

> Check: $f(-1) = 2(-1)^4 - 8(-1)^3 + 4(-1)^2 + 10(-1) - 4$
> $= 0$

(ii) $f(x) = 0$

$2(x^2 - 3x + 1)(x + 1)(x - 2) = 0$

$x = \frac{3 \pm \sqrt{9-4(1)(1)}}{2}$ or -1 or 2

$= \frac{3 \pm \sqrt{5}}{2}, -1, 2$

$\therefore$ **No. of real roots = 4**

(iii) $f(x) = 2(x^2 - 3x + 1)(x + 1)(x - 2)$

When $x = \frac{1}{2}$,

> Remainder Theorem:
> $f(x)$ divided by $(x - a) \Rightarrow$ remainder is $f(a)$

$f\left(\frac{1}{2}\right) = 2\left[\frac{1}{4} - \frac{3}{2} + 1\right]\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)$

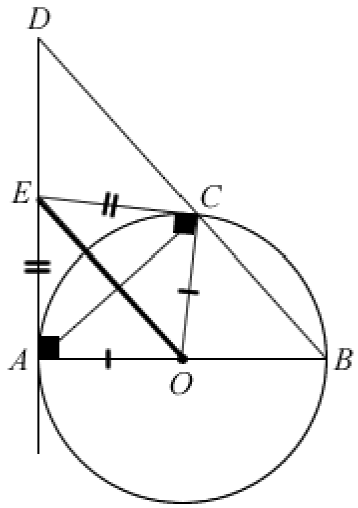
$= 2\left(-\frac{1}{4}\right)\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)$

$= 1\frac{1}{8}$

Remainder = $1\frac{1}{8}$

6. **Topic: Geometric Proofs**

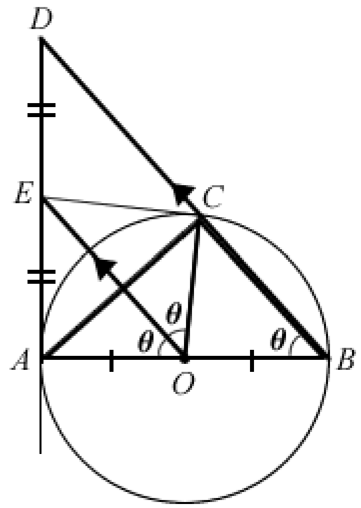
(i)



$AE = CE$ (Tangents from a common external point E)
$AO = CO$ (Radius of circle)
$\angle OAE = \angle OCE$
$= 90°$ (Radius $\perp$ Tangent)

$\therefore \Delta AEO \equiv \Delta CEO$ **(SAS)**

(ii)



Let $\angle AOE = \theta$.
$\angle COE = \theta$ ($\Delta AEO \equiv \Delta CEO$)
$\angle AOC = 2\theta$
$\Rightarrow \angle ABC = \theta$
($\angle$ at centre = $2 \times \angle$ at circumference)
$\Rightarrow EO \parallel DB$ ($\angle AOE = \angle ABC = \theta$ - corresponding $\angle$s)

$\therefore$ **By Midpoint Theorem,**
***E* is the mid-pt of *AD*.**

7. **Topic: Trigonometry (Trigonometric Functions)**

(i) Amplitude = **4**

(ii) Period = $\frac{360°}{2 \text{ cycles}}$ = **180°**

> $y = a \cos bx + c$: Amplitude = a
> Period = $\frac{360°}{b}$

(iii) Minimum point occurs when $\cos 2x = -1$

$\Rightarrow \quad 2x = \cos^{-1}(-1)$

$2x = 180°$

$x = 90°$

$\Rightarrow \quad y = 4(-1) - 2 = -6$

$\therefore$ **coordinates of the minimum point of the curve is (90°, –6)**

(iv) When $y = 0$, $4\cos 2x - 2 = 0$

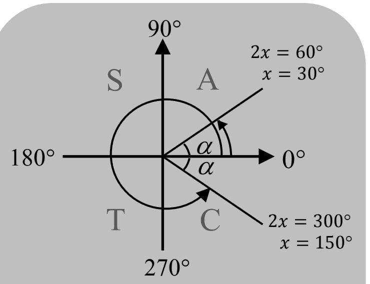
$4\cos 2x = 2$

$\cos 2x = \frac{1}{2}$

Basic angle $\alpha = 60°$

$2x = 60°, 300°$

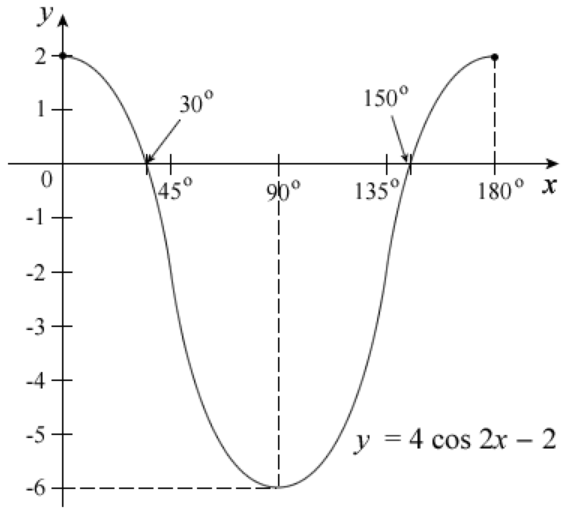
$x = 30°, 150°$



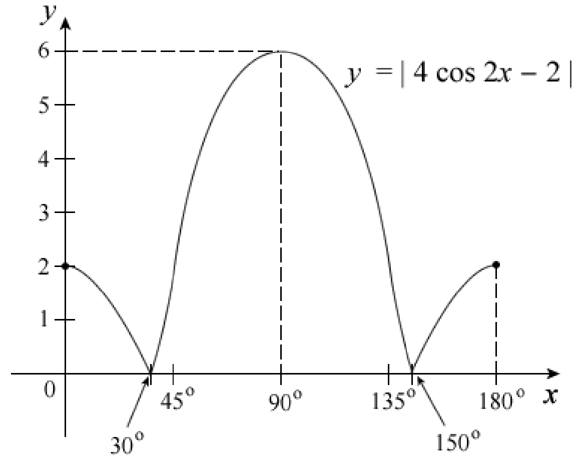
$\alpha = 60°$
$0° \le x \le 180°$
$0° \le 2x \le 360°$

$\therefore$ **coordinates where curve meets the x-axis are (30°, 0)and (150°, 0)**

(v)



(vi)



8. **Topic: Applications of Differentiation & Integration (Maxima & Minima, Area Under Curve)**

(i) $y = x^3 - ax + b$

$\frac{dy}{dx} = 3x^2 - a$

From the diagram, minimum point is (2, 0)

$\Rightarrow$ Sub $\frac{dy}{dx} = 0$ and $x = 2$ into $\frac{dy}{dx}$:

$3(2)^2 - a = 0$

$12 - a = 0$

$a = 12$

Sub (2, 0) into y: $0 = 2^3 - 12(2) + b$

$0 = 8 - 24 + b$

$b = 16$

$\therefore$ **$a = 12$, $b = 16$**

(ii) From (i),

$y = x^3 - 12x + 16$

$\frac{dy}{dx} = 3x^2 - 12$

At maximum point,

$\frac{dy}{dx} = 3x^2 - 12 = 0$

$3(x^2 - 4) = 0$

$3(x + 2)(x - 2) = 0$

$x = -2$ or 2 (rejected $\because$ max. point occurs when $x < 0$ in the diagram)

Sub $x = -2$ into y:

$y = (-2)^3 - 12(-2) + 16$

$= -8 + 24 + 16$

$= 32$

Check: $\frac{d^2y}{dx^2} = 6x$

Sub $x = -2$ into $\frac{d^2y}{dx^2}$:

$\frac{d^2y}{dx^2} = 6(-2) = -12 < 0$

$\Rightarrow$ (–2, 32) is a maximum point

$\therefore$ **coordinates of maximum point is (–2, 32).**

(iii) Area of shaded region $= \int_0^2 (x^3 - 12x + 16)\,dx$

$= \left[\frac{x^4}{4} - \frac{12x^2}{2} + 16x\right]_0^2$

$= \left[\frac{x^4}{4} - 6x^2 + 16x\right]_0^2$

$= [\frac{(2)^4}{4} - 6(2)^2 + 16(2)] - [\frac{(0)^4}{4} - 6(0)^2 + 16(0)]$

$= 4 - 24 + 32 - 0$

$=$ **12 units^2**

9. **Topic: Further Trigonometric Identities (R-Formula)**

(i) From the diagram,

$\angle OAD = \theta$ (corresponding $\angle$s)

$\sin\theta = \frac{OD}{4}$

$OD = 4\sin\theta$

$\angle ABC = 180° - 90° - \angle BAC$

$= 180° - 90° - (180° - 90° - \theta)$

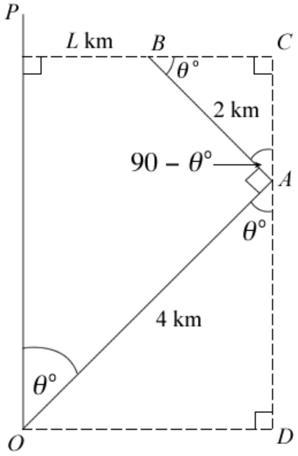
$= \theta$

$\cos\theta = \frac{BC}{2}$

$BC = 2\cos\theta$

$\therefore L = OD - BC$

$= \mathbf{4\sin\theta - 2\cos\theta}$ **(Shown)**



(ii) $4\sin\theta - 2\cos\theta = R\sin(\theta - \alpha)$

$= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$= R\cos\alpha\sin\theta - R\sin\alpha\cos\theta$

Addition Formula:
$\sin(A - B) = \sin A\cos B - \cos A\sin B$

Comparing coefficients,

$4 = R\cos\alpha$ ………. (1)

$2 = R\sin\alpha$ ………. (2)

$\frac{(2)}{(1)}$: $\frac{\sin\alpha}{\cos\alpha} = \frac{2}{4}$

$\tan\alpha = \frac{1}{2}$

$\alpha \approx 26.6°$

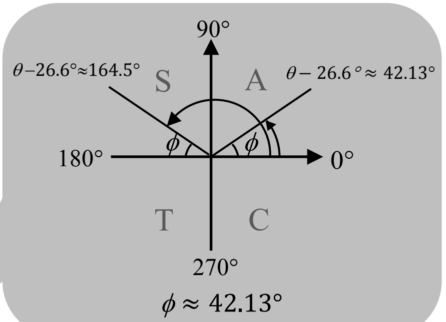
$(1)^2 + (2)^2$: $R^2\cos^2\alpha + R^2\sin^2\alpha = 4^2 + 2^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 16 + 4$

$R^2 = 20$

$R = \sqrt{20}$ or $-\sqrt{20}$ (rejected)

$\therefore$ $\boldsymbol{L = \sqrt{20}\sin(\theta - 26.6°)}$



(iii) When $L = 3$,

$\sqrt{20}\sin(\theta - 26.6°) = 3$

$\sin(\theta - 26.6°) = \frac{3}{\sqrt{20}}$

Basic angle $\phi \approx 42.13°$

$\theta - 26.6° \approx 42.13°, 180° - 42.13°$

$\theta \approx 68.7°, 164.5°$ (rejected $\because \theta$ is acute)

$\therefore$ $\boldsymbol{\theta \approx 68.7°}$ **(3 sig. fig.)**

10. **Topics: Coordinate Geometry, Integration, Applications of Differentiation (Rate of Change)**

(i) $\frac{dy}{dx} = \frac{6}{(2x-1)^2}$

Sub $x = 2$ into $\frac{dy}{dx}$:

Gradient of curve at $P(2, 9) = \frac{6}{3^2} = \frac{2}{3}$

$\Rightarrow$ Gradient of normal $= -\frac{3}{2}$

> Gradients m_1 and m_2 of two $\perp$ lines $\Leftrightarrow m_1 m_2 = -1$

Equation of normal to the curve at P:

$y - 9 = -\frac{3}{2}(x-2)$

$y = -\frac{3}{2}x + 3 + 9$

$y = -\frac{3}{2}x + 12$ .... (1)

> Equation of line with gradient m and point (x_1, y_1):
> $(y - y_1) = m(x - x_1)$

At $Q(0, y)$,

Sub $x = 0$ into (1): $y = -\frac{3}{2}(0) + 12$

$= 12$

$\Rightarrow Q = (0, 12)$

At $R(x, 0)$,

Sub $y = 0$ into (1): $-\frac{3}{2}x + 12 = 0$

$\frac{3}{2}x = 12$

$x = 8$

$\Rightarrow R = (8, 0)$

$\therefore$ Mid-point of $QR = \left(\frac{0+8}{2}, \frac{12+0}{2}\right)$

$=$ **(4, 6)**

> Midpoint of (x_1, y_1) and (x_2, y_2):
> $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

(ii) $y = \int \frac{6}{(2x-1)^2}\,dx$

$= \int 6(2x-1)^{-2}\,dx$

$= \frac{6(2x-1)^{-1}}{2(-1)} + c$

$= \frac{3(2x-1)^{-1}}{-1} + c$

$y = -\frac{3}{2x-1}$ .... (2)

> $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

Since $P(2, 9)$ lies on the curve, sub $x = 2$, $y = 9$ into (2):

$9 = -\frac{3}{2(2)-1} + c$

$9 = -1 + c$

$c = 10$

$\therefore$ $\mathbf{y = -\frac{3}{2x-1} + 10}$

(iii) x-coordinate increases at 0.03 units per second

$\Rightarrow \frac{dx}{dt} = 0.03$

Rate of change of y-coordinate:

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{6}{(2x-1)^2} \times 0.03$

> Chain Rule:
> $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

At $P(2, 9)$, sub $x = 2$ into $\frac{dy}{dt}$:

$\frac{dy}{dt} = \frac{6}{(2(2)-1)^2} \times 0.03$

$=$ 0.02 units per second

$\therefore$ ***y*-coordinate increases at 0.02 units per second.**

11. **Topics: Coordinate Geometry, Circles**

(i) Centre of C_1, $O = (0, 0)$

Radius of C_1, $OP = \sqrt{(0-8)^2 + (0+6)^2}$

$= 10$ units

$\therefore$ Equation of C_1: $(x-0)^2 + (y-0)^2 = 10^2$

$$\mathbf{x^2 + y^2 = 100}$$

> Length of Line Segment $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

> Equation of circle with centre (a, b) and radius r: $(x-a)^2 + (y-b)^2 = r^2$

(ii) Center of C_2, Q = Midpoint of OP

$= \left(\frac{0+8}{2}, \frac{0+(-6)}{2}\right)$

$= (4, -3)$

Radius of C_2, $QP = \sqrt{(4-8)^2 + (-3+6)^2}$

$= 5$ units

$\therefore$ Equation of C_2: $(x-4)^2 + (y+3)^2 = 5^2$

$x^2 - 8x + 16 + y^2 + 6y + 9 = 25$

$$\mathbf{x^2 + y^2 - 8x + 6y = 0}$$

> Midpoint of (x_1, y_1) and (x_2, y_2): $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

(iii) Gradient of $OP = \frac{0+6}{0-8} = \frac{3}{-4}$

$\Rightarrow$ Gradient of $AB = \frac{4}{3}$

> Gradients m_1 and m_2 of two $\perp$ lines $\Leftrightarrow m_1 m_2 = -1$

Using $Q(4, -3)$ from (ii):

Equation of AB: $y + 3 = \frac{4}{3}(x-4)$

$y = \frac{4}{3}x - \frac{25}{3}$ ……… (1)

> Equation of line with gradient m and point (x_1, y_1): $(y - y_1) = m(x - x_1)$

A and B also lie on C_1 with equation obtained in (i):

$x^2 + y^2 = 100$ ………….. (2)

Sub (1) into (2):

$x^2 + \left(\frac{4}{3}x - \frac{25}{3}\right)^2 = 100$

$x^2 + \frac{16x^2}{9} - \frac{200}{9}x + \frac{625}{9} = 100$

$\frac{25}{9}x^2 - \frac{200}{9}x + \frac{625}{9} = 0$

$25x^2 - 200x - 275 = 0$

$x^2 - 8x - 11 = 0$

$x = \frac{8 \pm \sqrt{64 - 4(1)(-11)}}{2(1)}$

$= \frac{8 \pm \sqrt{108}}{2}$

$= \frac{8 \pm 6\sqrt{3}}{2}$

$= 4 \pm 3\sqrt{3}$

$\therefore$ ***x*-coordinates of *A* and *B* are $4 + 3\sqrt{3}$ and $4 - 3\sqrt{3}$ respectively.**