ADDITIONAL MATHEMATICS — 4038/01

Paper 1 Suggested Solutions — October/November 2008

1. **Topic: Trigonometry (Trigonometric Ratios)**

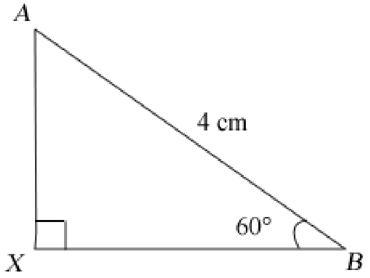
(i) $\angle ABX = 60°$

$\sin\angle ABX = \frac{AX}{AB}$

$\sin 60° = \frac{AX}{4}$

$AX = \frac{\sqrt{3}}{2} \times 4$

$= \mathbf{2\sqrt{3}\ cm}$



(ii) Using Pythagoras' theorem in ΔABX,

$AB^2 = AX^2 + BX^2$

$16 = (2\sqrt{3})^2 + BX^2$

$BX^2 = 16 - 12$

$= 4$

$BX = 2$ cm

In ΔAXC, $\tan\angle ACB = \frac{AX}{CX}$

$= \frac{2\sqrt{3}}{4}$

$\angle ACB = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **(Shown)**

> $AX = 2\sqrt{3}$ from (i)
> $CX = BC + BX$
> $= 2 + 2 = 4$ cm

2. **Topics: Indices, Simultaneous Equations**

$9^x(27)^y = 1$

$(3^{2x})(3^{3y}) = 3^0$

$3^{2x+3y} = 3^0$

Comparing the indices: $2x + 3y = 0$

$6y = -4x$ ………. (1)

$8^y \div (\sqrt{2})^x = 16\sqrt{2}$

$2^{3y} \div 2^{\frac{1}{2}x} = (2^4)\left(2^{\frac{1}{2}}\right)$

$2^{3y-\frac{1}{2}x} = 2^{4+\frac{1}{2}}$

Comparing the indices: $3y - \frac{1}{2}x = 4 + \frac{1}{2}$

$3y - \frac{1}{2}x = \frac{9}{2}$

$6y - x = 9$ ……..…… (2)

Sub (1) into (2): $-4x - x = 9$

$x = -\frac{9}{5}$

Sub $x = -\frac{9}{5}$ into (1): $6y = -4\left(-\frac{9}{5}\right)$

$= \frac{36}{5}$

$y = \frac{6}{5}$

$\therefore \boldsymbol{x = -\frac{9}{5},\ y = \frac{6}{5}}$

3. **Topic: Simultaneous Equations (Solution by Inverse Matrix Method)**

$\mathbf{A} = \begin{pmatrix} 7 & -8 \\ 1 & 6 \end{pmatrix}$

$\mathbf{A}^{-1} = \frac{1}{(7)(6)-(-8)(1)}\begin{pmatrix} 6 & 8 \\ -1 & 7 \end{pmatrix}$

$= \mathbf{\frac{1}{50}\begin{pmatrix} 6 & 8 \\ -1 & 7 \end{pmatrix}}$

> $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
>
> $\mathbf{M}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Given: $7q - 8p = 11$
$q + 6p = -7$

Expressing the above as a matrix equation,

$\begin{pmatrix} 7 & -8 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$

> $\mathbf{A}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$

Multiply both sides by $\mathbf{A}^{-1}$,

$\begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 6 & 8 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 11 \\ -7 \end{pmatrix}$

$= \frac{1}{50}\begin{pmatrix} 10 \\ -60 \end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} 1 \\ -6 \end{pmatrix}$

> $\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 11 \\ -7 \end{pmatrix}$
>
> $\mathbf{I}\begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 11 \\ -7 \end{pmatrix}$
>
> $\begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 11 \\ -7 \end{pmatrix}$

$\therefore\ \boldsymbol{q = \frac{1}{5}, p = -\frac{6}{5}}$

4. **Topic: Differentiation & Integration**

(i) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3\ln x) = x^3\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) + \ln x\frac{\mathrm{d}}{\mathrm{d}x}(x^3)$

$= x^3 \times \frac{1}{x} + 3x^2\ln x$

$= \boldsymbol{x^2 + 3x^2\ln x}$

> Product Rule:
> $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

(ii) $\int x^2\ln x\,\mathrm{d}x = \frac{1}{3}\int 3x^2\ln x\,\mathrm{d}x$

$= \frac{1}{3}\left[\int (3x^2\ln x + x^2 - x^2)\,\mathrm{d}x\right]$

$= \frac{1}{3}\left[\int (3x^2\ln x + x^2)\,\mathrm{d}x - \int x^2\,\mathrm{d}x\right]$

$= \frac{1}{3}\int (3x^2\ln x + x^2)\,\mathrm{d}x - \frac{1}{3}\int x^2\mathrm{d}x$

$= \boldsymbol{\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c}$

> From (i): $\frac{\mathrm{d}}{\mathrm{d}x}(x^3\ln x) = x^3 + 3x^2\ln x \Rightarrow$
> $\int (3x^2\ln x + x^2)\,\mathrm{d}x = x^3\ln x + c$

5. **Topics: Partial Fractions, Applications of Differentiation (Gradients)**

(i) $\frac{8x-46}{(x-5)(x+1)} = \frac{A}{(x-5)} + \frac{B}{(x+1)}$

Multiply both sides by $(x-5)(x+1)$:

$8x - 46 = A(x+1) + B(x-5)$

Let $x = -1$: $-8 - 46 = B(-6)$
$B = 9$

Let $x = 5$: $40 - 46 = A(6)$
$A = -1$

$\therefore\ \boldsymbol{\frac{8x-46}{(x-5)(x+1)} = -\frac{1}{(x-5)} + \frac{9}{(x+1)}}$

(ii) From (i), $y = \frac{8x-46}{(x-5)(x+1)}$

$= -\frac{1}{(x-5)} + \frac{9}{(x+1)}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = (x-5)^{-2} - 9(x+1)^{-2}$

When $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = (2-5)^{-2} - 9(2+1)^{-2} = \frac{1}{9} - \frac{9}{9}$

$= \boldsymbol{-\frac{8}{9}}$

6. **Topic: Applications of Differentiation & Integration (Kinematics)**

(i) $v = 6t - \frac{1}{2}t^2$

Since the cyclist is at rest, $v = 0$ at point B.

$\Rightarrow \quad 6t - \frac{1}{2}t^2 = 0$

$\frac{1}{2}t(12 - t) = 0$

$t = 0$ (reject) or 12

$\therefore$ Time taken from A to B = **12 s**

(ii) Distance $AB = \int_0^{12}\left(6t - \frac{1}{2}t^2\right)\mathrm{d}t$

$= \left[3t^2 - \frac{1}{6}t^3\right]_0^{12}$

$= 3(12)^2 - \frac{1}{6}(12)^3$

$=$ **144 m**

(iii) Acceleration $a = \frac{\mathrm{d}v}{\mathrm{d}t} = 6 - t$

When $t = 8$, $a = 6 - 8$

$=$ **–2 ms^{-2}**

7. **Topic: Applications of Differentiation (Gradients, Tangents & Normals)**

Given $y = \frac{\sin x}{2 - \cos x}$, $0 < x < \frac{\pi}{2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2 - \cos x)\cos x - \sin x \cdot \sin x}{(2 - \cos x)^2}$$

$$= \frac{2\cos x - \cos^2 x - \sin^2 x}{(2 - \cos x)^2}$$

$$= \frac{2\cos x - 1}{(2 - \cos x)^2}$$

Tangent to curve is parallel to the x-axis

$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 0$

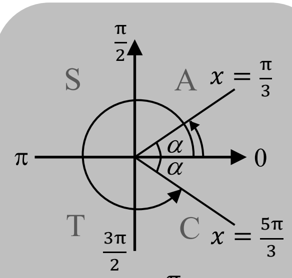
$\frac{2\cos x - 1}{(2 - \cos x)^2} = 0$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2} \Rightarrow$ Basic angle $\alpha = \frac{\pi}{3}$

$x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ (reject $\because 0 < x < \frac{\pi}{2}$)

$\therefore \boldsymbol{x = \frac{\pi}{3}}$



$\alpha = \frac{\pi}{3}$

8. **Topic: Further Trigonometric Identities**

(i) For $\sin 3x + \sin x = 4\sin x\cos^2 x$

L.H.S.: $\sin 3x + \sin x$

$= 2\sin\frac{3x+x}{2}\cos\frac{3x-x}{2}$

$= 2\sin 2x\cos x$

$= 2(2\sin x\cos x)\cos x$

$= 4\sin x\cos^2 x = \text{R.H.S.}$ (**Shown**)

Factor Formula:
$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

Double Angle Formula:
$\sin 2A = 2\sin A\cos A$

(ii) $\sin 3x + \sin x = 2\cos^2 x$

From (i): $\sin 3x + \sin x = 4\sin x\cos^2 x$

$4\sin x\cos^2 x = 2\cos^2 x$

$4\sin x\cos^2 x - 2\cos^2 x = 0$

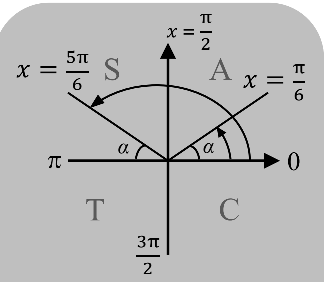
$2\cos^2 x(2\sin x - 1) = 0$

$\cos x = 0 \qquad \sin x = \frac{1}{2}$

$x = \frac{\pi}{2} \qquad$ Basic angle $\alpha = \frac{\pi}{6}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \boldsymbol{x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}$



$\alpha = \frac{\pi}{6}$

N.B.: Basic angle is +ve and acute.

9. **Topic: Simultaneous Equations**

Let Ann's age be x and Betty's age be y.

$x^2 - 2y^2 = 6(x-y)$ ………. (1)

$x + y = 5(x-y)$

$x + y = 5x - 5y$

$6y = 4x$

$y = \frac{2}{3}x$ ……………... (2)

$x^2 - 2y^2$: "… twice the square of Betty's age subtracted from the square of Ann's age"

$= 6(x-y)$: "… equal to 6 times the difference of their ages."

$x + y = 5(x-y)$: "… sum of their ages is equal to 5 times the difference of their ages"

Sub (2) into (1): $x^2 - 2\left(\frac{4}{9}x^2\right) = 6\left(x - \frac{2}{3}x\right)$

$\frac{1}{9}x^2 = 2x$

$x^2 - 18x = 0$

$x(x-18) = 0$

$x = 0$ (reject as Ann is older than Betty) or $x = 18$

When $x = 18$,

$y = \frac{2}{3}(18) = 12$

$\therefore$ **Ann is 18 years old and Betty is 12 years old.**

10. **Topic: Quadratic Equations & Inequalities**

(a) $ax^2 + 5x + 2 > 0$

$\Rightarrow b^2 - 4ac < 0$ (no real roots)

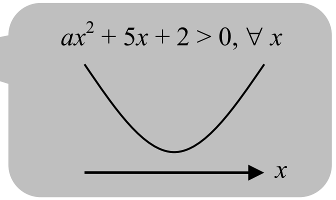
$25 - 4a(2) < 0$

$25 < 8a$

$a > \frac{25}{8}$

$> 3\frac{1}{8}$

$\therefore$ **smallest integer $a = 4$**



(b) $-5x^2 + bx - 2 < 0$

$b^2 - 4ac < 0$ (no real roots)

$b^2 - 4(-5)(-2) < 0$

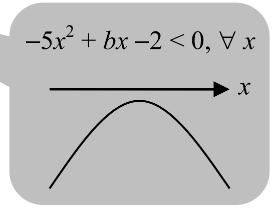
$b^2 - 40 < 0$

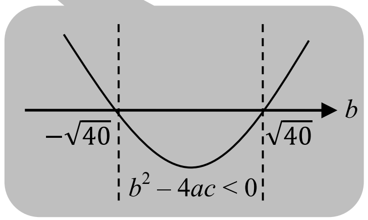
$(b - \sqrt{40})(b + \sqrt{40}) < 0$

$-\sqrt{40} < b < \sqrt{40}$

$\Rightarrow b > -\sqrt{40} \approx -6.32$

$\therefore$ **smallest integer $b = -6$**





11. **Topic: Binomial Expansions**

(i) $(r+1)^{th}$term of $\left(x+\frac{k}{x}\right)^7 = \binom{7}{r}\ x^{7-r}\ (kx^{-1})^r$

$= \binom{7}{r} k^r x^{7-2r}$

In expanding $(a+b)^n$, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Coefficient of $x^3 = \binom{7}{2} k^2$

$= 21k^2$

$x^3 = x^{7-2r} \Rightarrow r = 2 \rightarrow T_2$

Coefficient of $x = \binom{7}{3} k^3$

$= 35k^3$

$x^1 = x^{7-2r} \Rightarrow r = 3 \rightarrow T_3$

Equating coefficients of x^3 and x,

$21k^2 = 35k^3$

$3k^2 = 5k^3$

$5k^3 - 3k^2 = 0$

$k^2(5k-3) = 0$

$k = 0$ (reject) or $k = \frac{3}{5}$

$\therefore \boldsymbol{k = \frac{3}{5}}$

(ii) $(1-5x^2)(x+\frac{k}{x})^7 = (1-5x^2)\left[\binom{7}{0} x^7 + \binom{7}{1} kx^5 + \cdots\right]$

$= x^7 - 35kx^7 + \cdots$

$\therefore$ coefficient of $x^7 = 1 - 35k$

$= 1 - 35\left(\frac{3}{5}\right)$

$= 1 - 7(3)$

$= \mathbf{-20}$

Terms beyond the 1st 2 terms of $(x+\frac{k}{x})^7$ are ignored as they do not form x^7 terms when multiplied by $(1-5x^2)$.

12. **Topic: Linear Law**

(i) Given $y = kb^x$,

$\lg y = \lg(kb^x)$

$\lg y = \lg k + x \lg b$

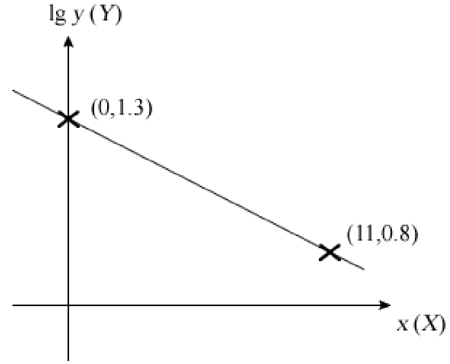
$\lg y = (\lg b)x + \lg k$

Letting Y be $\lg y$ and X be x, the graph of $\lg y$ against $\lg x$ is a straight line

$Y = (\lg b)X + \lg k$

$= mX + c$

From the graph,



Gradient m:

$\Rightarrow \quad \lg b = \frac{1.3-0.8}{0-11}$

$= -\frac{1}{22}$

$b = 10^{-\frac{1}{22}}$

$\approx$ **0.90 (2 sig. fig.)**

When $X = 0$, Y-intercept $= 1.3$

$\Rightarrow \quad \lg k = 1.3$

$k = 10^{1.3}$

$\approx$ **20 (2 sig. fig.)**

(ii) From (i), $y = 10^{1.3}\left(10^{-\frac{1}{22}}\right)^x$

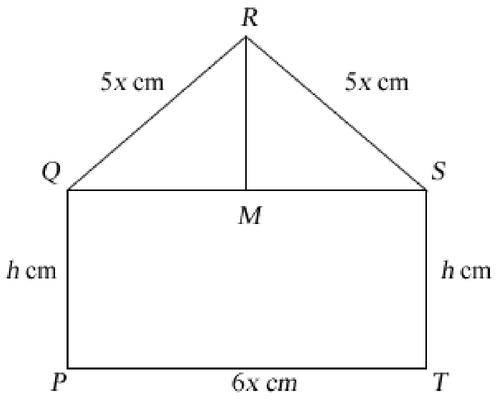
When $x = 8$,

$y = 10^{1.3}\left(10^{-\frac{1}{22}}\right)^8$

$\approx \mathbf{8.64}$

13. **Topic: Applications of Differentiation (Maxima & Minima)**

(i)



Perimeter of the window = 360cm (given)

$$5x \times 2 + h \times 2 + 6x = 360$$
$$5x + h + 3x = 180$$
$$h = 180 - 8x$$

Let *RM* be the height of ΔQRS.
⇒ *M* is the mid-point of *QS*. (ΔQRS is isosceles)

Using Pythgoras' theorem,

$$RM = \sqrt{(5x)^2 - (3x)^2}$$
$$= 4x \text{ cm}$$

$$\therefore A = \frac{1}{2}(6x)(4x) + (h)(6x)$$
$$= 12x^2 + (180 - 8x)6x$$
$$= \mathbf{1080x - 36x^2} \textbf{ (Shown)}$$

Area A = Area of ΔQRS +
Area of rectangle *PQST*

To show that $A = 1080x - 36x^2$, express all unknowns i.e. h and RM (height of ΔQRS) in terms of x.

(ii) $\frac{dA}{dx} = 1080 - 72x$

When $\frac{dA}{dx} = 0$,

$$72x = 1080$$
$$x = 15 \text{ cm}$$

When $x = 15$ cm,

$$A = 1080(15) - 36(15)^2$$
$$= \mathbf{8100cm^2}$$

(iii) $\frac{d^2A}{dx^2} = -72$

$\Rightarrow \frac{d^2y}{dx^2} < 0$

⇒ **The stationary value of *A* is a maximum.**