MATHEMATICS (H2) 9740/02
Paper 2 Suggested Solutions October/November 2010

1. **Topic:Complex Numbers (Complex Roots of Quadratic Equations)**

(i)
$$\begin{aligned} x^2 - 6x + 34 &= 0 \\ x &= \frac{6\pm\sqrt{36-4(34)}}{2} \\ &= \frac{6\pm\sqrt{-100}}{2} \\ &= \frac{6\pm10\sqrt{-1}}{2} \\ &= \frac{6\pm10\mathrm{i}}{2} \\ &= 3 \pm 5\mathrm{i} \end{aligned}$$

$\sqrt{-1} = \mathrm{i}$

$\therefore$ 3 + 5i and 3 − 5i are the solutions.

(ii)
$$x^4 + 4x^3 + x^2 + ax + b = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since $x = -2 + \mathrm{i}$ is a root, by Factor Theorem,

$$\begin{aligned} (-2+\mathrm{i})^4 + 4(-2+\mathrm{i})^3 + (-2+\mathrm{i})^2 + a(-2+\mathrm{i}) + b &= 0 \\ -7 - 24\mathrm{i} + 4(-2+11\mathrm{i}) + 3 - 4\mathrm{i} - 2a + a\mathrm{i} + b &= 0 \\ -7 - 24\mathrm{i} - 8 + 44\mathrm{i} + 3 - 4\mathrm{i} - 2a + a\mathrm{i} + b &= 0 \\ -12 - 2a + b + (16+a)\,\mathrm{i} &= 0 \end{aligned}$$

Factor Theorem:
$x-a$ is a factor of $f(x) \Leftrightarrow f(a) = 0$

Both real and imaginary parts must be 0

$16 + a = 0$ $\qquad -12 - 2a + b = 0$

$a = -16$ $\qquad b = 2a + 12 = -20$

$\therefore$ ***a* = −16 and *b* = −20**

Since $x = -2 + \mathrm{i}$ is a root and the coefficient of each term in (1) is real, then by Complex Conjugate Root Theorem, $x = -2 - \mathrm{i}$ is also a root.

$$\begin{aligned} \Rightarrow [x-(-2+\mathrm{i})]\,[x-(-2-\mathrm{i})] &= (x+2-\mathrm{i})(x+2+\mathrm{i}) \\ &= (x+2)^2 - \mathrm{i}^2 \\ &= x^2 + 4x + 4 + 1 \\ &= x^2 + 4x + 5 \end{aligned}$$

Factorizing (1),

$$\begin{aligned} x^4 + 4x^3 + x^2 - 16x - 20 &= 0 \\ (x^2 + 4x + 5)(x^2 - 4) &= 0 \\ (x^2 + 4x + 5)(x-2)(x+2) &= 0 \\ x &= -2 + \mathrm{i}, -2 - \mathrm{i}, 2, -2 \end{aligned}$$

$\therefore$ **The other roots are −2 – i, −2 and 2.**

2. **Topic: Series (Mathematical Induction, Method of Difference)**

(i) Let P_n be the statement

$$\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7), n \in \mathbb{Z}^+.$$

When $n = 1$,

$$\begin{aligned} \text{L.H.S.} &= \sum_{r=1}^{1} r(r+2) \\ &= 1(1+2) \\ &= 3 \\ \text{R.H.S.} &= \frac{1}{6}(1)(1+1)(2+7) \\ &= \frac{2\times9}{6} \\ &= 3 \\ \text{L.H.S.} &= \text{R.H.S.} \end{aligned}$$

$\therefore$Since L.H.S. = R.H.S., P_1 is true

Assume P_k is true for some $k \in \mathbb{Z}^+$ i.e.

$$\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$$

To show that P_{k+1} is also true i.e.

$$\sum_{r=1}^{k+1} r(r+2) = \frac{1}{6}(k+1)(k+2)[2(k+1)+7],$$

$(k+1)^{\text{th}}$ term

$$\begin{aligned}
\text{L. H. S.} &= \sum_{r=1}^{k+1} r(r+2) \\
&= \sum_{r=1}^{k} r(r+2) + \underbrace{(k+1)(k+3)}_{} \\
&= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3) \\
&= \frac{1}{6}(k+1)[2k^2+7k+6k+18] \\
&= \frac{1}{6}(k+1)[2k^2+13k+18] \\
&= \frac{1}{6}(k+1)(2k+9)(k+2) \\
&= \frac{1}{6}(k+1)(k+2)[2(k+1)+7] \\
&= \text{R. H. S.}
\end{aligned}$$

Bring out common factor $\frac{1}{6}(k+1)$ since it appears on RHS.

$\therefore$ Since L. H. S. = R. H. S., P_{k+1} is true if P_k is true.

Since P_1 is true and P_{k+1} if P_k is true, P_n is true $\forall n \geq 1$, $n \in \mathbb{Z}^+$ by mathematical induction.

From MF15: Partial fractions decomposition (Non-repeated linear factors):

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

(ii) (a) Let $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2} \Rightarrow A(r+2) + Br = 1$

$r = 0 \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

$r = -2 \Rightarrow -2B = 1 \Rightarrow B = -\frac{1}{2}$

$\Rightarrow \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$

Cover-up Rule may be used directly to save time.

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{r(r+2)} &= \sum_{r=1}^{n}\left[\frac{1}{2r} - \frac{1}{2(r+2)}\right] \\
&= \frac{1}{2}\left[\sum_{r=1}^{n}\left[\frac{1}{r} - \frac{1}{(r+2)}\right]\right] \\
&= \frac{1}{2}\Big[\quad \frac{1}{1} \quad - \cancel{\frac{1}{3}} \\
&= \quad + \frac{1}{2} \quad - \cancel{\frac{1}{4}} \\
&= \quad \cancel{+\frac{1}{3}} \quad - \cancel{\frac{1}{5}} \\
&= \quad \cancel{+\frac{1}{4}} \quad - \cancel{\frac{1}{6}} \\
&= \qquad \vdots \\
&= \quad \cancel{+\frac{1}{n-2}} \quad - \cancel{\frac{1}{n}} \\
&= \quad \cancel{+\frac{1}{n-1}} \quad - \frac{1}{n+1} \\
&= \quad \cancel{+\frac{1}{n}} \quad - \frac{1}{n+2}\Big] \\
&= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] \\
&= \mathbf{\frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}} \textbf{ (shown)}
\end{aligned}$$

(b)
$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{1}{r(r+2)} &= \lim_{n\to\infty} \sum_{r=1}^{n}\left[\frac{1}{r(r+2)}\right] \\
&= \lim_{n\to\infty}\left[\frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right]
\end{aligned}$$

Using expression proven in 2(ii)(a)

As $n \to \infty$, $\frac{1}{2(n+1)} \to 0$ and $\frac{1}{2(n+2)} \to 0$

$\Rightarrow \lim_{n\to\infty}\left[\frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right] = \frac{3}{4} - 0 - 0 = \frac{3}{4}$

$\therefore \sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \mathbf{\frac{3}{4}}$

Since it converges to a constant value (with a sum to infinity of $\frac{3}{4}$), this is a convergent series.

3. **Topic: Differentiation**

(i) Given

$$\begin{aligned} y &= x\sqrt{x+2} \\ &= x(x+2)^{\frac{1}{2}} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (x+2)^{\frac{1}{2}} + x\left(\frac{1}{2}\right)(x+2)^{-\frac{1}{2}} \\ &= \sqrt{x+2} + \frac{x}{2\sqrt{x+2}} \\ &= \frac{2(x+2)+x}{2\sqrt{x+2}} \\ &= \frac{3x+4}{2\sqrt{x+2}} \end{aligned}$$

Product Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

When

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{3x+4}{2\sqrt{x+2}} &= 0 \\ \boldsymbol{x} &= \boldsymbol{-\frac{4}{3}} \end{aligned}$$

$\therefore$ **There is only one stationary point when $\boldsymbol{x = -\frac{4}{3}}$.**

(ii) (a) Given

$$\begin{aligned} y^2 &= x^2(x+2) \\ y &= \pm x\sqrt{x+2} \end{aligned}$$

From Part (i), we have

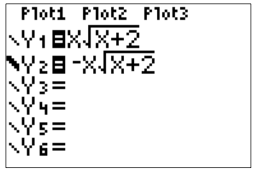
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x+4}{2\sqrt{x+2}}$$

When $x = 0$,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{4}{2\sqrt{2}} \\ &= \pm\frac{2}{\sqrt{2}} \\ &= \pm\sqrt{2} \end{aligned}$$

$\therefore$ **Possible values of the gradient is $\boldsymbol{\sqrt{2}}$ and $\boldsymbol{-\sqrt{2}}$.**

(b) Using G. C. (refer to Appendix for detailed steps),



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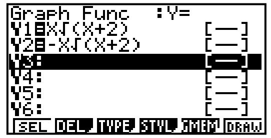

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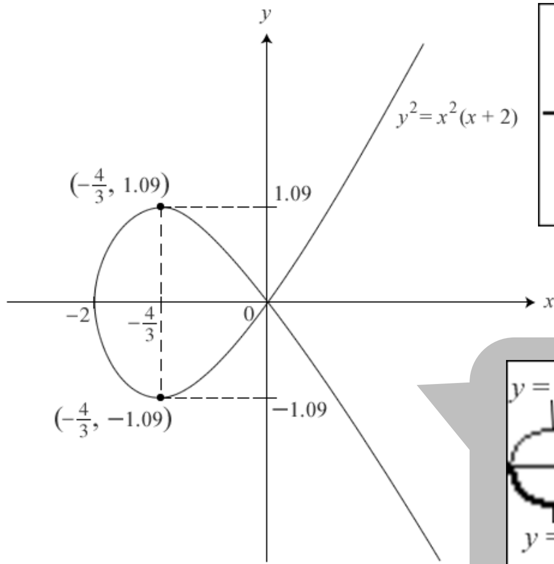
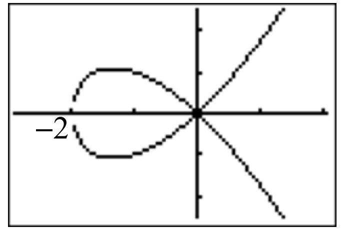
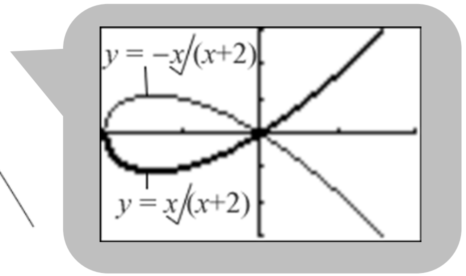
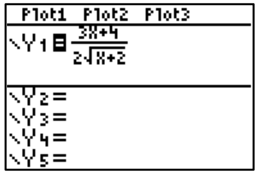






(iii) Using G. C. (refer to Appendix for detailed steps),



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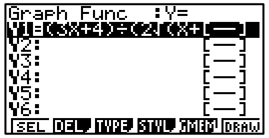

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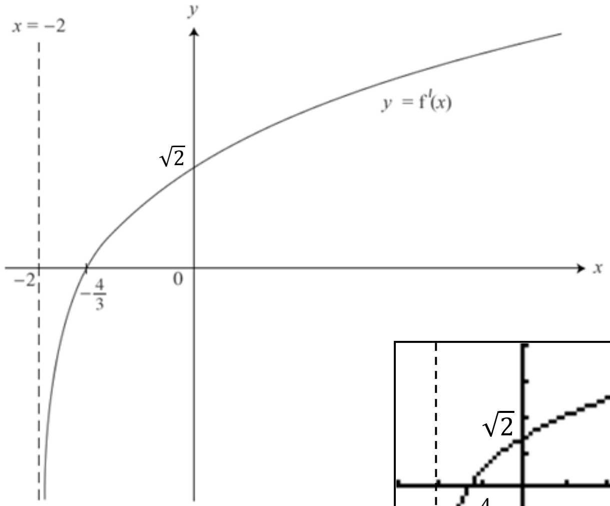
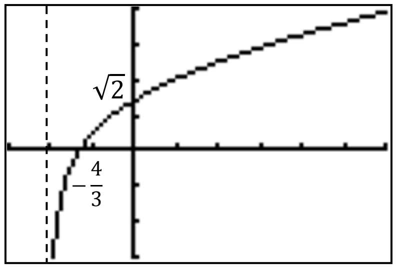
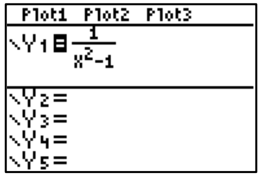




$x = -2$

4. **Topic: Functions**

(i) Using G. C. (refer to Appendix for detailed steps),



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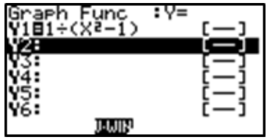

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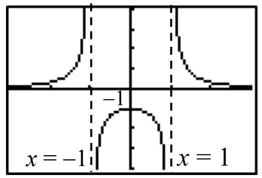
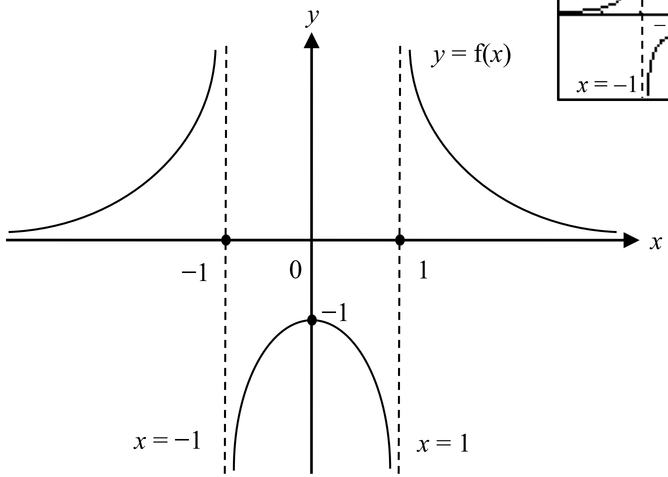


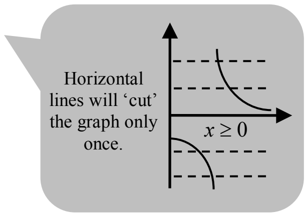
When $x = 0$, $y = -1$



(ii) For the function f^{-1} to exist, f must be one-one over the given domain, where there exists only one value of x for each image of f.

From the sketch in Part (i), f is one-one when $x \geq 0, x \neq 1$.

Hence the least value of k is 0.



(iii) $\text{fg}(x) = \text{f}[\text{g}(x)] = \text{f}\left(\frac{1}{x-3}\right)$

$$= \frac{1}{\frac{1}{(x-3)^2}-1}$$

$$= \frac{(x-3)^2}{1-(x-3)^2}$$

$$= \frac{(x-3)^2}{1-(x^2-6x+9)}$$

$$= \frac{(x-3)^2}{-x^2+6x-8}$$

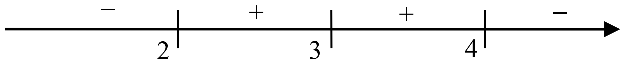
$$= \frac{(x-3)^2}{(4-x)(x-2)} \textbf{ (shown)}$$

(iv) $\text{fg}(x) > 0$

$$\frac{(x-3)^2}{(4-x)(x-2)} > 0$$

Since $(x-3)^2 \geq 0, (4-x)(x-2)$ must be positive,

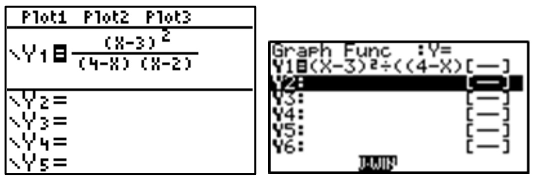
By number line:



$\therefore \mathbf{2 < x < 4, x \neq 3}$

ALTERNATIVE APPROACH

Using G. C. to plot fg(x) (refer to Appendix for detailed steps),



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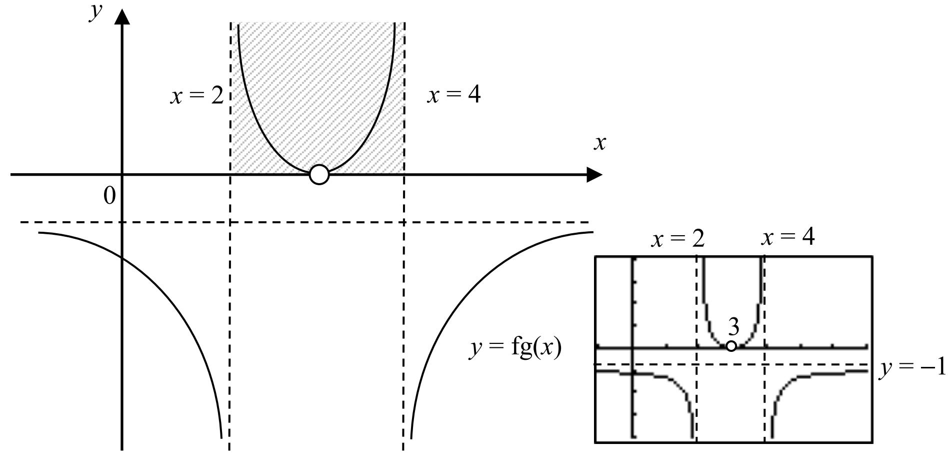


From the graph, $\mathbf{2 < x < 4, x \neq 3}$ for $\text{fg}(x) > 0$

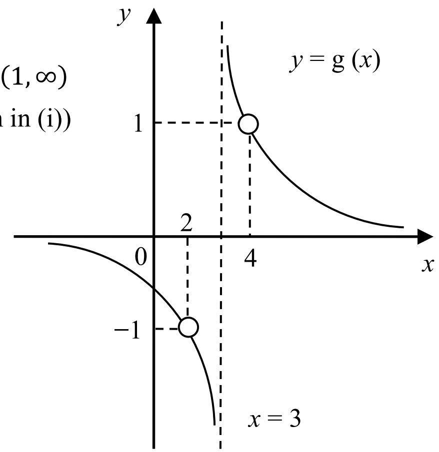
(v) Given $\text{g}(x) = \frac{1}{(x-3)}, x \in \mathbb{R}, x \neq 2, x \neq 3, x \neq 4$. From graph of g($x$),

$R_g = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$.

$\Rightarrow$ When $D_f = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$

$R_f = (-\infty, -1) \cup (0, \infty)$ (from graph in (i))

$\therefore$ **Range of fg(x)** $= (-\infty, \mathbf{-1}) \cup (\mathbf{0}, \infty)$



ALTERNATIVE APPROACH

From the graph sketched in part (iv), **range of fg(x)** $= (-\infty, \mathbf{-1}) \cup (\mathbf{0}, \infty)$.

5. **Topic: Sampling**

(i) **Due to the great cultural and regional variety of the international spectators, it would be difficult to divide them into appropriate the strata for a suitable analysis.**

Moreover, given the large size, multinational and mobile nature of the population of spectators, it will be tedious and time-consuming to accurately obtain the required representative sample 1% of spectators in each stratum.

(ii) **A systematic sample of 1% of the spectators could be obtained by first randomly interviewing a person leaving the premise of the catering facilities, and thereafter interviewing every 100th person leaving the premise of the catering facilities.**

Stratified Sampling:

Divide population into mutually-exclusive subgroups (strata), and then apply random or systematic sampling within each subgroup.

Systematic Sampling:

To obtain a systematic sample of size n from a population of size N, pick a random element from among the first $k = \frac{N}{n}$ elements, and thereafter picking every k^{th} element.

6. **Topic: Hypothesis Testing**

Given:

$$n = 11$$
$$\sum t = 454.3$$
$$\sum t^2 = 18778.43$$

Unbiased estimate of population mean, $\bar{t} = \frac{\sum t}{n} = \frac{454.3}{11} = \mathbf{41.3}$

Unbiased estimate of population variance, $s^2 = \frac{1}{n-1}\left[\sum t^2 - \frac{(\sum t)^2}{n}\right]$

From MF15

$$= \frac{1}{10}\left[18778.43 - \frac{(454.3)^2}{11}\right]$$
$$\approx \mathbf{1.584}$$

Let μ be the mean time required by an employee to complete a task.

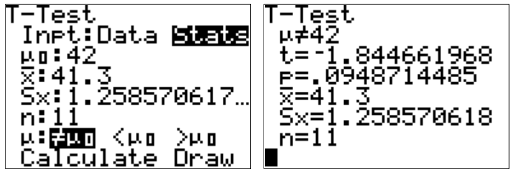
To test H_o: $\mu = 42.0$ against
H_1 : $\mu \neq 42.0$ at 10% of significance

Testing for change in μ
$\Rightarrow$ Two-tailed test

Reject H_o if p-value < 0.10.

Applying t-test with $\bar{t} = 41.3$, $n = 11$, $s^2 = 1.584$ using G. C. (refer to Appendix for detailed steps),

Since population variance is not given and sample size is small (n < 30), a t-test is used.



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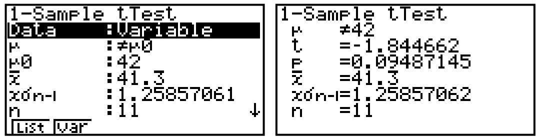

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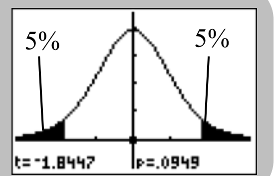


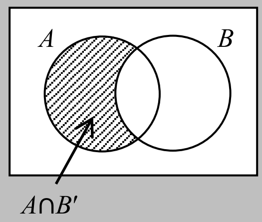
From GC, the p-value = 0.09487 < 0.10, we reject H_o.

Hence, there is sufficient evidence at the 10% significance level that there has been a change in the mean time required by an employee to complete the task.

7. **Topic: Probability**

Given P(A) = 0.7, P(B) = 0.6, P($A|B'$) = 0.8

(i)
$$\begin{aligned} P(A|B') &= 0.8 \\ \frac{P(A\cap B')}{P(B')} &= 0.8 \\ P(A\cap B') &= 0.8 \times P(B') \\ &= 0.8 \times [1 - P(B)] \\ &= \mathbf{0.32} \end{aligned}$$



(ii)
$$\begin{aligned} P(A\cup B) &= P(A\cap B') + P(B) \\ &= 0.32 + 0.6 \\ &= \mathbf{0.92} \end{aligned}$$

(iii)
$$\begin{aligned} P(B'|A) &= \frac{P(B'\cap A)}{P(A)} \\ &= \frac{0.32}{0.7} \\ &= \mathbf{\frac{16}{35}} \end{aligned}$$

Given P(C) = 0.5 and A and C are independent.

(iv)
$$\begin{aligned} P(A'\cap C) &= P(C) - P(A\cap C) \\ &= P(C) - P(A)\,P(C) \\ &= 0.5 - 0.7\times 0.5 \\ &= \mathbf{0.15} \end{aligned}$$

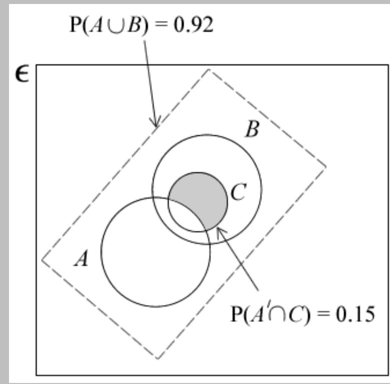
A and C are independent
$\Rightarrow P(A \cap C) = P(A)\,P(C)$

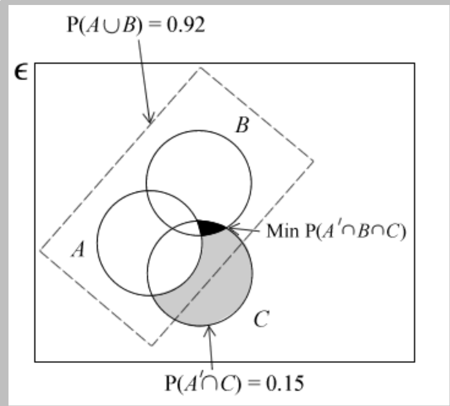
Note: This also means
$$\begin{aligned} P(C) - P(A)\,P(C) &= [1 - P(A)]\,P(C) \\ \Rightarrow P(A' \cap C) &= P(A')\,P(C) \end{aligned}$$

(v) Max P($A' \cap B \cap C$) case (C subset of B):

When $C \subseteq B \Rightarrow A' \cap B \cap C \subseteq A' \cap C$

$$\begin{aligned} \therefore P(A' \cap B \cap C) &\le P(A' \cap C) \\ P(A' \cap B \cap C) &\le 0.15 \end{aligned}$$





Min P($A' \cap B \cap C$) case (minimal intersection between B and C):

$$\begin{aligned} P(A \cup B \cup C) &\le 1 \\ P(A \cup B) + P(A' \cup B' \cap C) &\le 1 \\ P(A \cup B) + [P(A' \cap C) - P(A' \cap B \cap C)] &\le 1 \\ 0.92 + 0.15 - P(A' \cap B \cap C) &\le 1 \\ P(A' \cap B \cap C) &\ge 0.92 + 0.15 - 1 \\ P(A' \cap B \cap C) &\ge 0.07 \end{aligned}$$

$$\therefore \mathbf{0.07 \le P(A' \cap B \cap C) \le 0.15}$$

This is just a 1-mark question. It should be sufficient to state either P($A' \cap B \cap C$) ≤ 0.15 or P($A' \cap B \cap C$) ≥ 0.07.

8. **Topic: Probability**

(i)

1st digit	2nd digit	3rd digit	4th digit	5th digit
3 ways (i.e. 3, 4, 5)	4 ways	3 ways	2 ways	1 way

$$\text{P(number is greater than 30,000)} = \frac{3\times4!}{5!} = \mathbf{\frac{3}{5}}$$

$$\text{P}(A) = \frac{\text{no.of ways for event } A \text{ to occur}}{\text{total no.of possible outcomes}}$$

(ii)

1st digit	2nd digit	3rd digit	4th digit	5th digit
3 ways (i.e. 1, 3, 5)	2 ways	1 way	2 ways (i.e. 2, 4)	1 way (i.e. 4 or 2)

$$\text{P(last 2 digits are both even)} = \frac{3!\times2!}{5!} = \mathbf{\frac{1}{10}}$$

(iii) Case 1 (1st digit is 3 or 5):

1st digit	2nd digit	3rd digit	4th digit	5th digit
2 ways (i.e. 3, 5)	3 ways	2 ways	1 way	2 ways (i.e. 1, 5 or 3)

Case 2 (1st digit is 4):

1st digit	2nd digit	3rd digit	4th digit	5th digit
1 way (i.e. 4)	3 ways	2 ways	1 way	3 ways (i.e. 1, 3, 5)

$$\text{P(number is greater than 30,000 and odd)} = \frac{2\times3!\times2+1\times3!\times3}{5!} = \mathbf{0.35}$$

9. **Topic: Normal Distribution**

Let X and Y be the random variables such that Ken makes X minutes of peak-rate and Y minutes of cheap-rate telephone calls, respectively, over a 3-month period.

Given $X \sim \text{N}(180, 30^2)$ $Y \sim \text{N}(400, 60^2)$

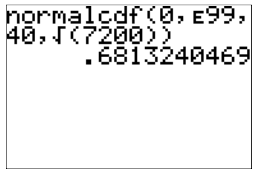
(i) $\text{E}(Y-2X) = \text{E}(Y) - 2\text{E}(X) = 400 - 2(180) = 40$

$\text{Var}(Y-2X) = \text{Var}(Y) + 2^2\text{Var}(X) = 60^2 + 4\times30^2 = 7200$

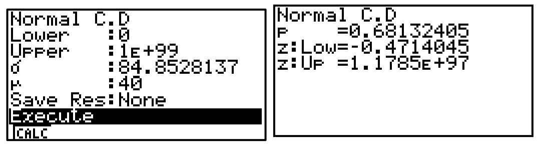
If $X \sim \text{N}(\mu_X, \sigma_X{}^2)$ and $Y \sim \text{N}(\mu_Y, \sigma_Y{}^2)$ are two independent normal distributions,
$aX \pm bY \sim \text{N}(a\mu_X \pm b\mu_Y, a^2\sigma_X{}^2 + b^2\sigma_Y{}^2)$

$\therefore Y-2X \sim \text{N}(40, 7200)$

Using G. C. (refer to Appendix for steps to access the normal distribution functions),



TI-84 Plus



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$\text{P}(Y > 2X) = \text{P}(Y - 2X > 0) = 0.68132 \approx \mathbf{0.681\ (3\ sig.fig.)}$

(ii) Let T be the random variable for the total cost (in dollars) of Ken's calls made over a three-month period $\Rightarrow$ $T = 0.12X + 0.05Y$

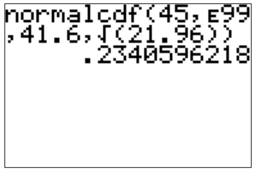
$$\begin{aligned}\text{E}(T) = \text{E}(0.12X + 0.05Y) &= 0.12\text{E}(X) + 0.05\text{E}(Y)\\ &= 0.12(180) + 0.05(400)\\ &= 41.6\end{aligned}$$

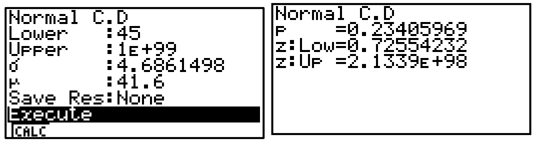
$$\begin{aligned}\text{Var}(T) = \text{Var}(0.12X + 0.05Y) &= 0.12^2\,\text{Var}(X) + 0.05^2\text{Var}(Y)\\ &= \underbrace{0.12^2\times30^2 + 0.05^2\times60^2}\\ &= 21.96\end{aligned}$$

$\therefore T \sim \text{N}(41.6, 21.96)$

Remember to square the 0.12 & 0.05 when calculating the variance!

Using G. C. (refer to Appendix for steps to access the normal distribution functions),





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$P(T > 45) = 0.23406 \approx$ **0.234 (3 sig.fig)**

(iii) Let W be the random variable for the total cost (in dollars) of Ken's peak-rate calls made over two three-month periods (X_1 and X_2 being the number of peak-rates calls in each period, respectively)

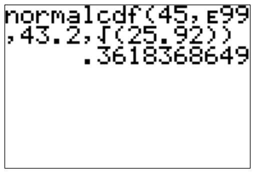
$$\Rightarrow \quad W = 0.12X_1 + 0.12X_2$$
$$E(W) = 2(0.12)\,E(X)$$
$$= 43.2$$
$$\text{Var}(W) = 2(0.12^2)\,\text{Var}(X)$$
$$= 25.92$$

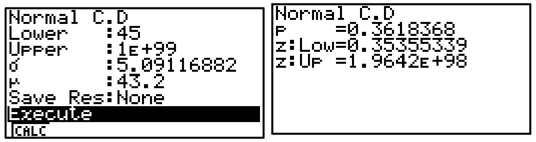
$\therefore W \sim N(43.2, 25.92)$

If X_1 and X_2 are two independent observations of the random variable X where $X \sim N(\mu, \sigma^2)$,

$aX_1 + aX_2 \sim N(2a\mu, 2a^2\sigma^2)$

Using G. C. (refer to Appendix for steps to access the normal distribution functions),





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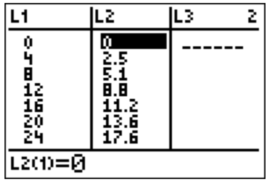
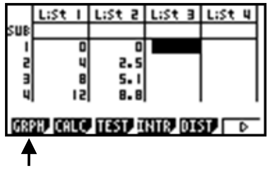
$P(W > 45) = 0.36183 \approx$ **0.362 (3 sig.fig)**

10. **Topic: Correlation and Regression**

(i) Using G. C. (refer to Appendix for detailed steps),





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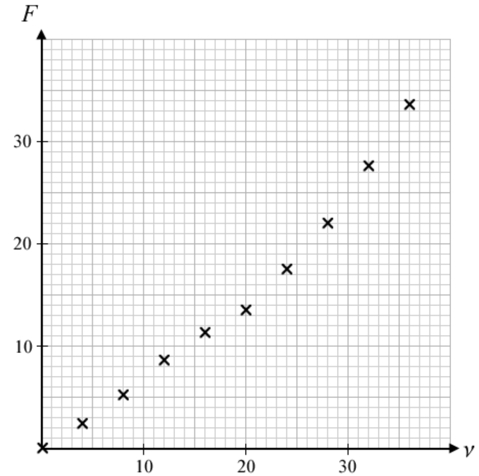
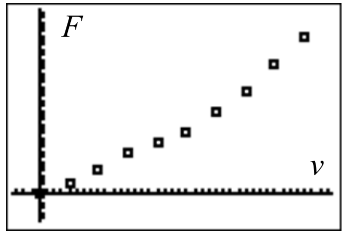
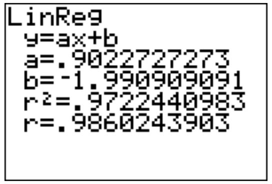
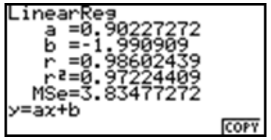




(ii) (a) For $F = a + bv$, using G. C. (refer to Appendix for detailed steps)





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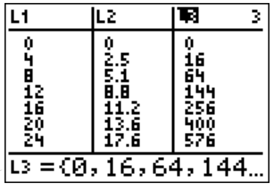
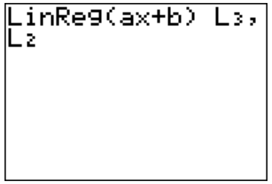
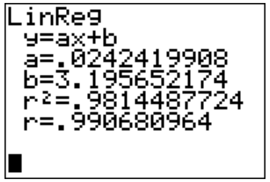
$r = 0.98602 \approx$ **0.9860 (4 decimal places)**

(b) For $F = c + dv^2$, , using G. C. (refer to Appendix for detailed steps)

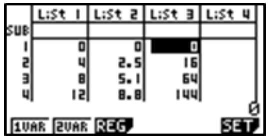


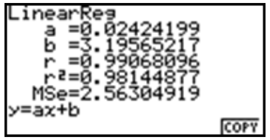


L3 = (L1)^2



TI-84 Plus





fx-9860G

$r = 0.99068 \approx$ **0.9907 (4 decimal places)**

(iii) Since the scatter diagram reveals a non-linear relationship between F and v in part (i) and the correlation coefficient between v^2 and F yields a higher value of 0.9907 (as compared to 0.9860 for v and F) in part (ii), $F = c + dv^2$ is a better model.

(iv) Sub a = 0.02424 and b = 3.1957 obtained by G. C. in part (ii)(b) into d and c respectively in $F = c + dv^2$,

$\Rightarrow$ Required equation $\boldsymbol{F = 3.1957 + 0.02424v^2}$

Sub F = 26.0,

$$26.0 = 3.1957 + 0.02424v^2$$
$$v = \mathbf{30.7}$$

Note: Reg ($ax + b$) used in G. C.

As the wind speed is controlled, *v* is the independent variable and we are using the regression line F on v^2 to predict *v*.

Since *F* is not the independent variable, we should not use the regression line of *v* on *F* or v^2 on *F* to estimate *v*.

11. **Topic: Binomial, Poisson Distributions& Their Normal Approximation**

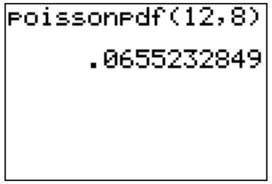
Let X be the random variable for the number of calls received in one minute. $X \sim \text{Po}(3)$.

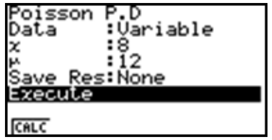
(i) Let X_4 be the random variable for the number of telephone calls received in a period of 4 minutes.

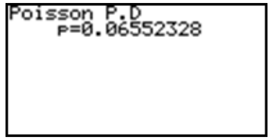
$\Rightarrow X_4 = 4X \sim \text{Po}(4\times3) \Rightarrow X_4 \sim \text{Po}(12)$

Additive Property of Poisson Distributions:
$$\sum_{i=1}^{n} X_i \sim \text{Po}\left(\sum_{i=1}^{n} \lambda_i\right)$$

Using G. C. (refer to Appendix for detailed steps),







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$P(X_4 = 8) = 0.06552 \approx$ **0.0655 (3 sig.fig)**

(ii) Let n be the number of minutes and X_n be the random variable for the number of telephone calls received in a period of n minutes.

$X_n \sim \text{Po}(3n)$

$$\begin{aligned} P(X_n = 0) &= 0.2 \\ e^{-3n}\frac{(3n)^0}{0!} &= 0.2 \\ e^{-3n} &= 0.2 \\ -3n &= \ln 0.2 \\ n &= -\tfrac{1}{3}\ln 0.2 \\ &= 0.53648 \text{ mins} \\ &= 32.188 \text{ seconds} \approx \textbf{32 seconds (nearest second)} \end{aligned}$$

Note: 0! = 1!

Probability density function of X, where $X \sim \text{Po}(\lambda)$:
$P(X = x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$

(iii) 12 hrs = 12 × 60 = 720 min

Let X_{720} be the random variable for the number of telephone calls received in 720 min.

$\Rightarrow \quad X_{720} = 720X \sim \text{Po}(720\times3) \Rightarrow X_{720} \sim \text{Po}(2160)$

Additive Property of Poisson Distributions

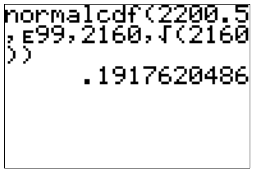
Since λ is large (>10), we use a normal distribution to approximate the Poisson distribution as follows

$\therefore X_{720} \sim N(2160, 2160)$ approximately.

When $\lambda > 10$,
$X \sim \text{Po}(\lambda) \approx N(\lambda, \lambda)$

$P(X_{720} > 2200) \to P(X_{720} > 2200.5)$ by continuity correction

Using G. C. (refer to Appendix for detailed steps),



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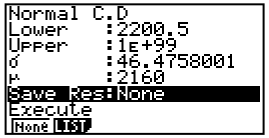

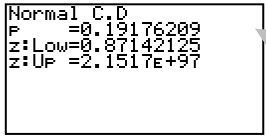

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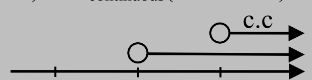
$P(X_{720} > 2200.5) = 0.19176 \approx$ **0.192 (3 sig.fig)**

Approximating a discrete distribution with a continuous distribution by Continuity Correction:

$P_{\text{discrete}}(X>x) \to P_{\text{continuous}}(X>x+0.5)$

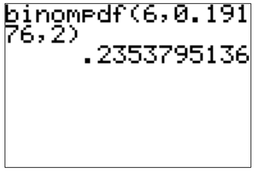


(iv) Let Y be the random variable for the number of busy working days out of 6 working days.

$Y \sim B(6, 0.19176)$

Binomial Distribution:
$X \sim B(n, p)$ where n = no. of trials = 6
p = probability of success
= 0.192 from part (iii)

Using G. C. (refer to Appendix for detailed steps),



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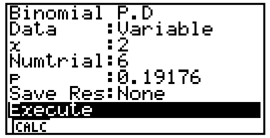
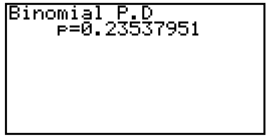




fx-9860G

$P(Y = 2) = 0.23537 \approx$ **0.235 (3 sig.fig)**

(v) Let W be the random variable for the number of busy working days out of 30 randomly chosen working days.

$W \sim B(30, 0.19176)$

$np = 30 \times 0.19176 = 5.7528 > 5$

$nq = 30 \times (1 - 0.19176) = 24.2472 > 5$

Since $np > 5$ and $nq > 5$, we use a normal distribution to approximate the Binomial distribution as follows

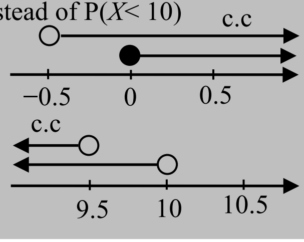
When n is large and $np >$ 5 and $nq >$ 5, $X \sim N(n, p) \approx N(np, npq)$

$W \sim N(5.7528, 5.7528 \times (1 - 0.19176))$

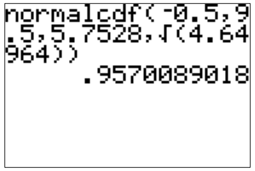
$\Rightarrow W \sim N(5.7528, 4.64964)$ approximately.

$P(0 \le W < 10) \rightarrow P(-0.5 < W < 9.5)$ by continuity correction.

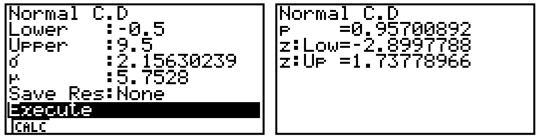
Since the number of days cannot be < 0, it should be $P(0 \le X < 10)$ instead of $P(X < 10)$



Using G. C. (refer to Appendix for detailed steps),



TI-84 Plus



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$P(-0.5 < W < 9.5) = 0.95700$

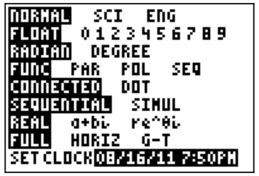
$\approx$ **0.957 (3 sig.fig)**

Appendix: Detailed G. C. Steps (for those still trapped in G. C. limbo)

Q3 (b)(ii), Q3 (iii), Q4 (i): Graph Sketching

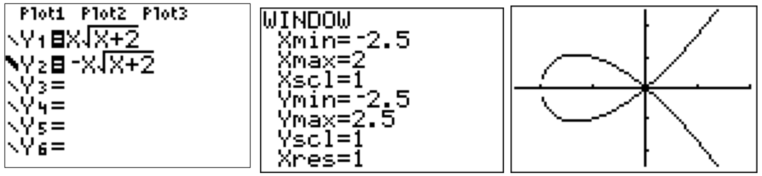
TI-84 Plus MODE

→ Ensure G. C. is in **FUNC** mode.

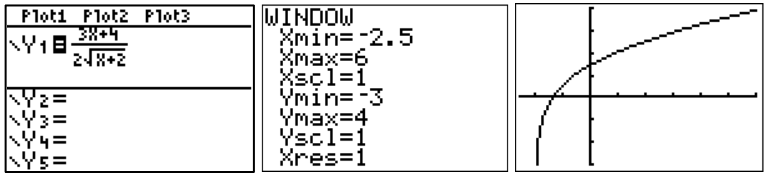


Y= WINDOW GRAPH

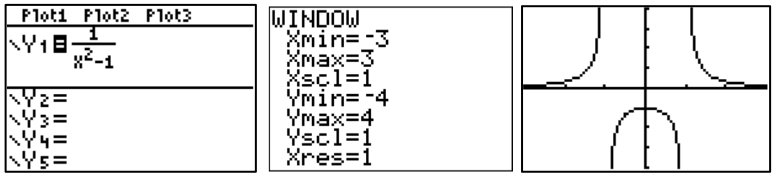
3(ii)(b)



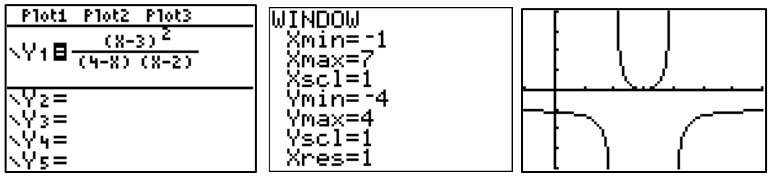
3(iii)



4(i)



4(iv)



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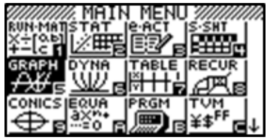
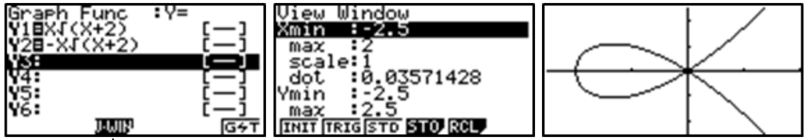
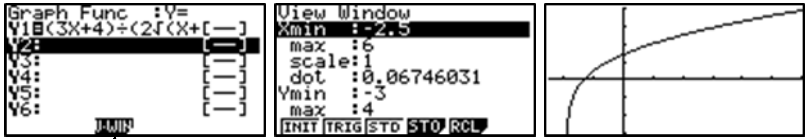


3(ii)(b)



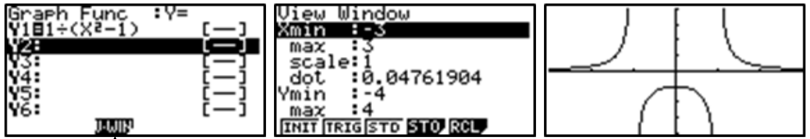
[SHIFT] ↑ [EXIT] [F6]

3(iii)



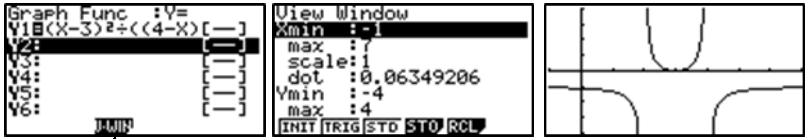
[SHIFT] ↑ [EXIT] [F6]

4(i)



[SHIFT] ↑ [EXIT] [F6]

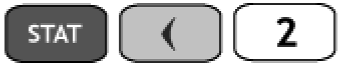
4(iv)



[SHIFT] ↑ [EXIT] [F6]

Q6: Hypothesis Testing (*t*-Test with Data Summary)

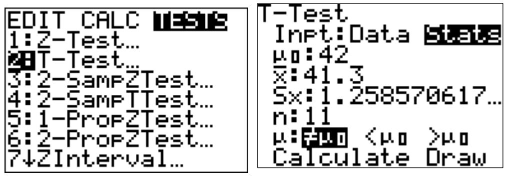
TI-84 Plus



→Select **Stats** input method.

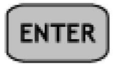
→ Enter the population/sample mean,√(sample variance), sample size.

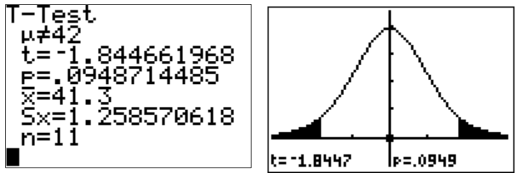
→Select μ:**$\neq\mu_o$** for two-tailed test.



→ Select **Calculate** for results in summary view.

→ Select **Draw** for results in graphical view.





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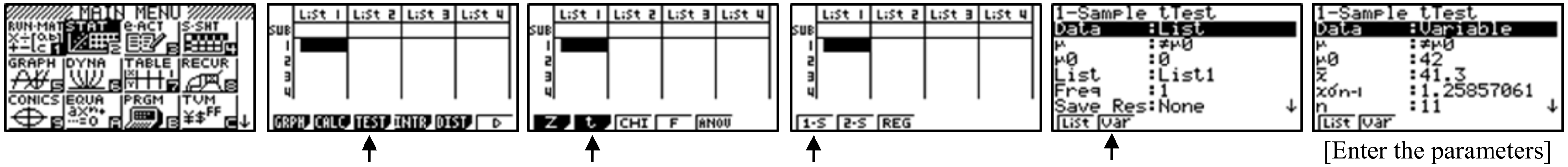
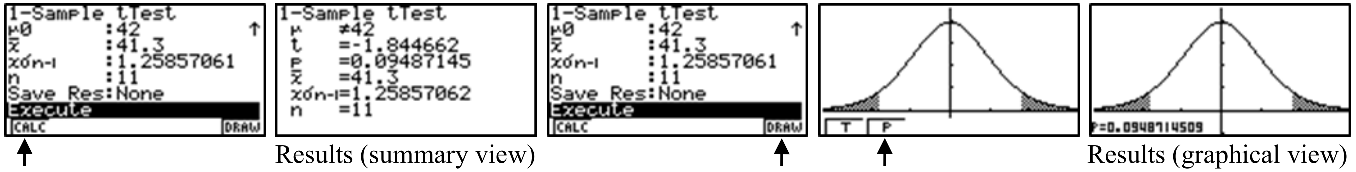


[Enter the parameters]



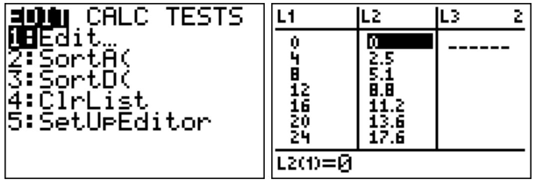
Results (summary view) Results (graphical view)

Q10 (i): Plotting Scatter Diagram

TI-84 Plus

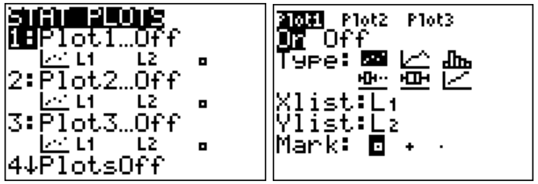
STAT ENTER

→ Enter *v* and *F* values in L1 and L2 respectively

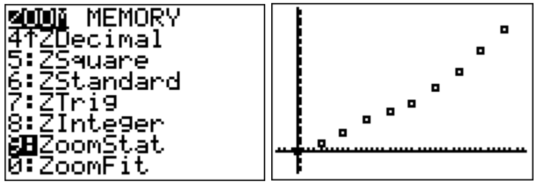


2nd Y= ENTER

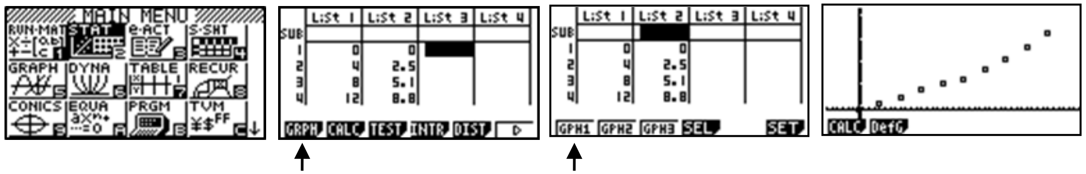
→ Turn On Plot1



ZOOM 9

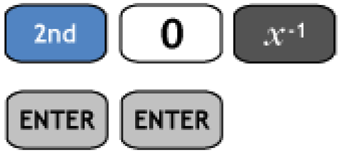


fx-9860G

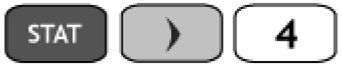


Q10 (ii)(a): Finding Correlation Coefficient

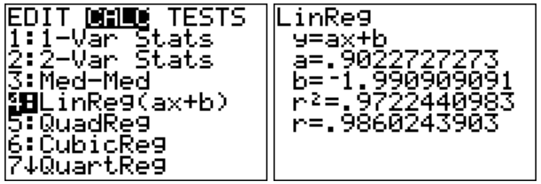
TI-84 Plus



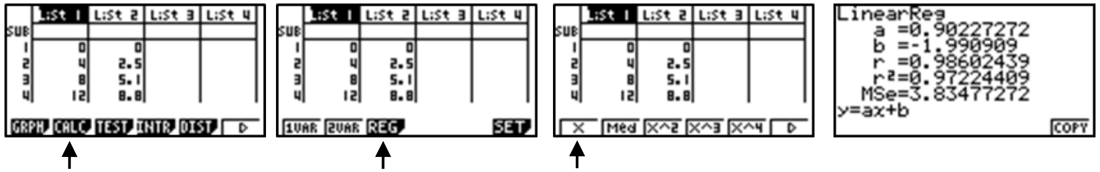
Note: The r value will not appear if you miss this step!



Note: L1 contains the values of v (independent variable) and L2 the values of F(dependent variable) aspopulated in 10(a).



fx-9860G



Q10 (ii)(b): Finding Correlation Coefficient

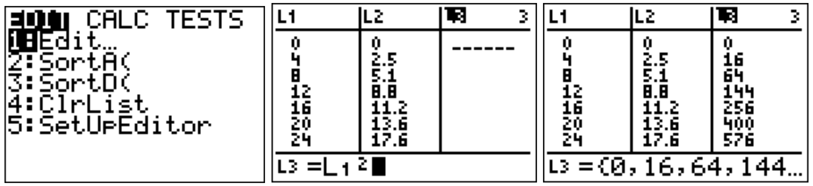
TI-84 Plus

STAT ENTER

→ Populate L3 with the square of values of v contained in L1.

2nd 1 x^2 ENTER

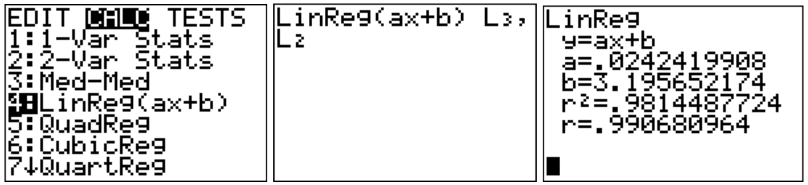
N.B. Make sure L3 is highlighted when doing this!



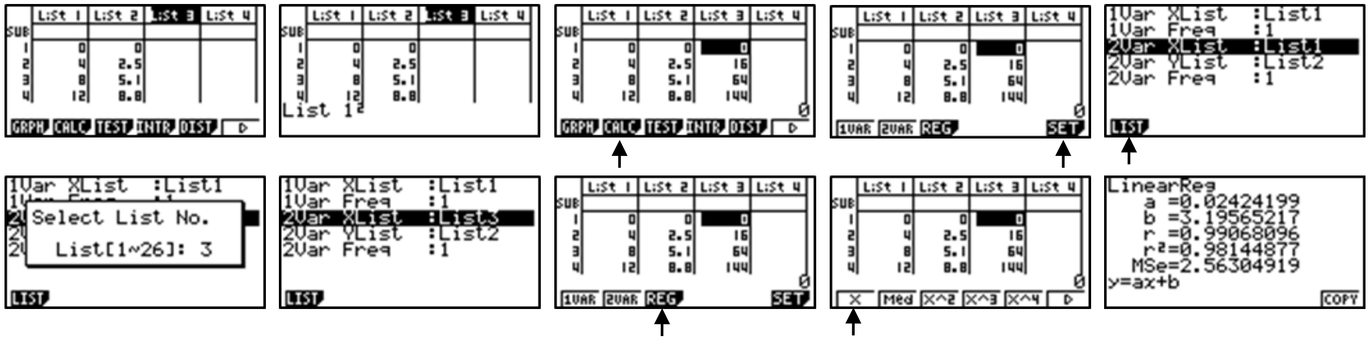
STAT) 4

2nd 3 , 2nd 2 ENTER

N.B. L3 contains the values of v^2 and L2 the values of F. The first argument contains the independent variable.



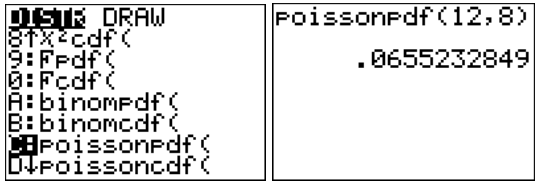
fx-9860G



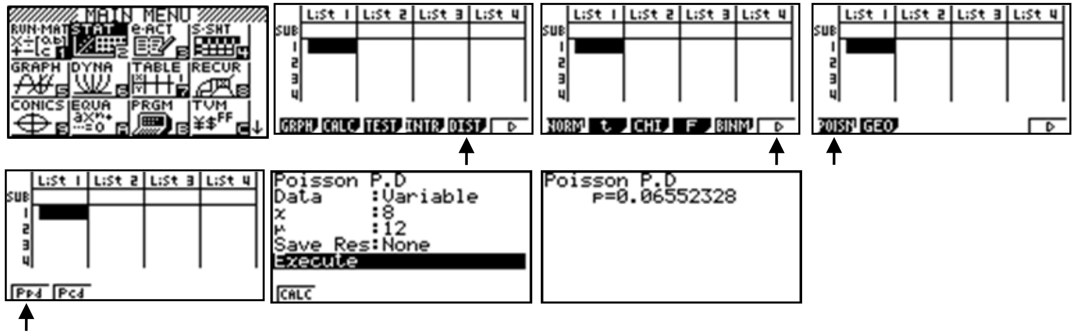
Q11 (i): Poisson Distribution

TI-84 Plus 2nd VARS ALPHA PRGM

→ Key in the parameters.



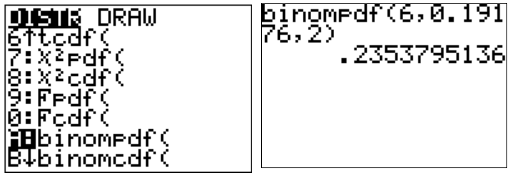
fx-9860G



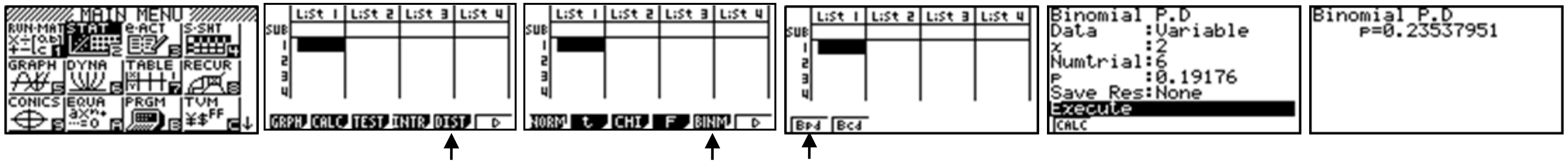
Q11 (iv): Binomial Distribution

TI-84 Plus 2nd VARS ALPHA MATH

→ Key in the parameters.



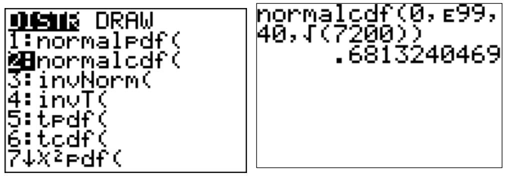
fx-9860G



Q9 (i-iii), Q11 (iii), Q11 (v): Normal Distribution

TI-84 Plus 2nd VARS 2

→ Key in the relevant parameters. Results shown are for Q9(i)



fx-9860G