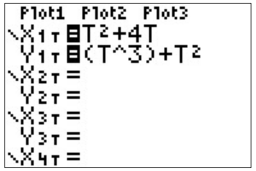
MATHEMATICS (H2) 9740/02

Paper 2 Suggested Solutions — October/November 2009

1. **Topic: Graphing Techniques**

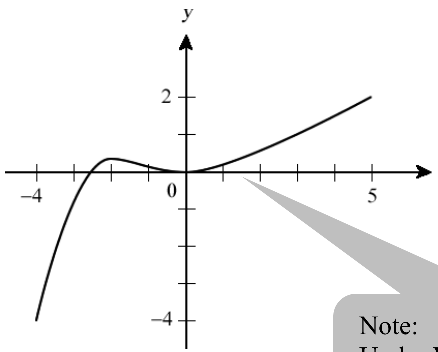
C: $x = t^2 + 4t$ ………………. (1)
$y = t^3 + t^2$ ………………. (2)

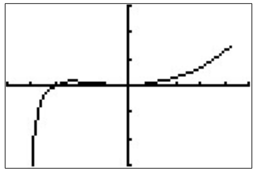
(i) When $t = -2$, $x = -4$, $y = -4$
When $t = 0$, $x = 0$, $y = 0$
When $t = 1$, $x = 5$, $y = 2$



TI-84 Plus

Casio fx-9860G





Note:
Under V-Window of Casio GC, set $T_{min} = -2$ and $T_{max} = 1$

(ii) $\frac{dx}{dt} = 2t + 4$

$\frac{dy}{dt} = 3t^2 + 2t$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$= \frac{3t^2+2t}{2t+4}$

When $t = 2$,

$\frac{dy}{dx} = \frac{3(2)^2+2(2)}{2(2)+4}$

$= 2$

and $x = 12$, $y = 12 \Rightarrow P(12, 12)$

$\therefore$ Equation of l:

$y - 12 = 2(x - 12)$

$\mathbf{y = 2x - 12}$ ………………. (3)

(iii) Given tangent l meets at Q.

Sub (1) and (2) into (3):

$t^3 + t^2 = 2(t^2 + 4t) - 12$

$t^3 - t^2 - 8t + 12 = 0$

$(t - 2)(t^2 + t - 6) = 0$ — Using Factor Theorem

$(t - 2)(t + 3)(t - 2) = 0$

$\therefore t = 2$ (reject) or $t = -3$

Reject as this belongs to point P

$\therefore$ When $t = -3$,

$x = (-3)^2 + 4(-3)$

$= -3$

$y = (-3)^3 + (-3)^2$

$= -18$

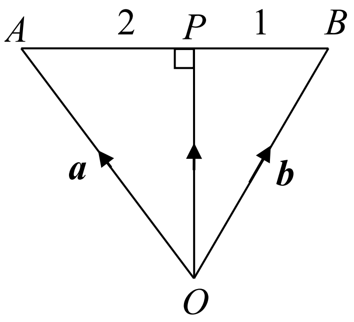
$\therefore$ **Coordinates of Q is $(-3, -18)$**

2. **Topic: Vectors**

Given $\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix}14\\14\\14\end{pmatrix}$ and $\overrightarrow{OB} = \mathbf{b} = \begin{pmatrix}11\\-13\\2\end{pmatrix}$.

(i) Given P divides AB in the ratio 2 : 1,

By ratio theorem:



$$\overrightarrow{OP} = \frac{1}{3}[\mathbf{a} + 2\mathbf{b}]$$
$$= \frac{1}{3}\left[\begin{pmatrix}14\\14\\14\end{pmatrix} + 2\begin{pmatrix}11\\-13\\2\end{pmatrix}\right]$$
$$= \begin{pmatrix}12\\-4\\6\end{pmatrix}$$

$\therefore$ **Coordinates of P is (12, −4, 6)**

(ii)
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \begin{pmatrix}11\\-13\\2\end{pmatrix} - \begin{pmatrix}14\\14\\14\end{pmatrix}$$
$$= \begin{pmatrix}-3\\-27\\-12\end{pmatrix}$$
$$\overrightarrow{AB}\cdot\overrightarrow{OP} = \begin{pmatrix}-3\\-27\\-12\end{pmatrix}.\begin{pmatrix}12\\-4\\6\end{pmatrix}$$
$$= -36 + 108 - 72$$
$$= 0$$

Two non-zero vector **a** and **b** are ⊥ if $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 90° = 0$

$\therefore \overrightarrow{AB}\cdot\overrightarrow{OP} = 0 \Rightarrow AB \perp OP$ **(Shown)**

(iii) Given **c** is a unit vector of $\overrightarrow{OP}$.

$$|\overrightarrow{OP}| = \sqrt{12^2 + (-4)^2 + 6^2}$$
$$= 14$$
$$\therefore \mathbf{c} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|}$$
$$= \begin{pmatrix}\frac{12}{14}\\-\frac{4}{14}\\\frac{6}{14}\end{pmatrix}$$
$$= \begin{pmatrix}\frac{6}{7}\\-\frac{2}{7}\\\frac{3}{7}\end{pmatrix}$$

The geometrical meaning of $|\mathbf{a}.\mathbf{c}|$ is length of projection of **a** onto $\overrightarrow{OP}$.

(iv)
$$\mathbf{a}\times\mathbf{p} = \begin{pmatrix}14\\14\\14\end{pmatrix}\times\begin{pmatrix}12\\-4\\6\end{pmatrix}$$
$$= \begin{pmatrix}(14\times6)+(4\times14)\\-(14\times6-14\times12)\\14\times(-4)-14\times12\end{pmatrix}$$
$$= \begin{pmatrix}140\\84\\-224\end{pmatrix}$$

The geometrical meaning of $|\mathbf{a}\times\mathbf{p}|$ is area of parallelogram

$\therefore$ Area of $\Delta\, OAP$

$= \frac{1}{2}|\mathbf{a}\times\mathbf{p}|$

$= \frac{1}{2}\left|\begin{pmatrix}140\\84\\-224\end{pmatrix}\right|$

$= \frac{1}{2}\sqrt{140^2+84^2+(-224)^2}$

$= \frac{1}{2}\sqrt{76832}$

$= \mathbf{98\sqrt{2}}$ **units^2**

3. **Topic: Functions**

Given $f(x) = \frac{ax}{bx-a}$, for $x \in \mathbb{R}$, $x \neq \frac{a}{b}$, $ab \neq 0$

(i) Let $y = \frac{ax}{bx-a}$ where $y = f(x)$

$$byx - ay = ax$$
$$(by - a)x = ay$$
$$x = \frac{ay}{by-a}$$
$$\therefore f^{-1}(x) = \frac{ax}{bx-a}$$
$$\therefore f^{-1}(x) = f(x)$$
$$\Rightarrow x = f(f(x))$$
$$\Rightarrow f^2(x) = x$$

$\therefore$ **Range of $f^2(x)$ is $R_{f^2} \in \mathbb{R} / \left\{\frac{a}{b}\right\}$**

(ii) Given $g(x) = \frac{1}{x}$ for $x \in \mathbb{R}$, $x \neq 0$

$\therefore R_g = R/\{0\}$

and $D_f = R/\left\{\frac{a}{b}\right\}$ where $a, b \neq 0$

$\therefore$ **fg does not exist because $R_g \not\subseteq D_f$**

(iii) Given $f^{-1}(x) = x$

From (1), we have $\frac{ax}{bx-a} = x$

$$\left(\frac{a}{bx-a} - 1\right)x = 0$$

$\therefore x = 0$ or $\frac{a}{bx-a} - 1 = 0$

$$a = bx - a$$
$$x = \frac{2a}{b}$$

$\therefore$ The solutions are: $\boldsymbol{x = 0}$ **or** $\boldsymbol{x = \frac{2a}{b}}$

4. **Topics: Differentiation, Differential Equations**

(i) $\frac{d^2n}{dt^2} = 10 - 6t$

$\frac{dn}{dt} = 10t - 3t^2 + c_1$, where c_1 is a constant

$n = 5t^2 - t^3 + c_1t + c_2$, where c_2 is a constant

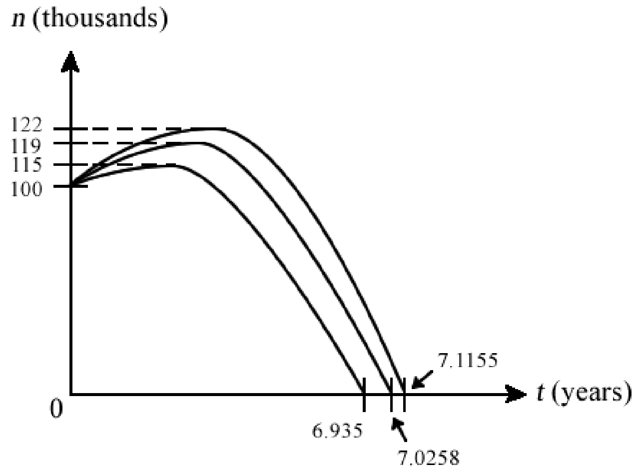
$\therefore$ Given $n = 100$ when $t = 0$,

$$100 = 5(0) - 0 + c_1(0) + c_2$$
$$c_2 = 100$$
$$\therefore n = 5t^2 - t^3 + c_1t + 100$$

When $c_1 = 1$, $n = 5t^2 - t^3 + 100 + t$, turning pt = (3.43, 122)

When $c_1 = 0$, $n = 5t^2 - t^3 + 100$, turning pt = (3.33, 119)

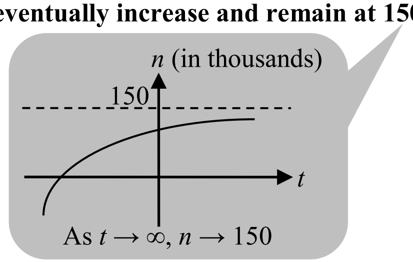
When $c_1 = -1$, $n = 5t^2 - t^3 + 100 - t$, turning pt = (3.23, 115)



(ii) Given by 2nd scientist:

$$\frac{dn}{dt} = 3 - 0.02n$$
$$\int \frac{1}{3-0.02n}\,dn = \int dt$$
$$\frac{1}{-0.02}\ln|3 - 0.02n| = t + c_3$$
$$\ln|3 - 0.02n| = -0.02t - 0.02c_3, \text{ where } c_3 \text{ is a constant}$$
$$3 - 0.02n = e^{-0.02t}e^{-0.02c_3}$$
$$0.02n = 3 - e^{-0.02t}e^{-0.02c_3}$$
$$n = 150 - 50e^{-0.02t}e^{-0.02c_3}$$
$$\boldsymbol{n = 150 - Ae^{-0.02t}}, \text{ where } A \text{ is a constant}$$

The population will eventually increase and remain at 150 000.



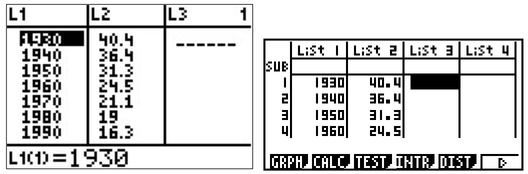
5. **Topic: Sampling**

A quota sample of 100 cinema- goers may be obtained by instructing the interviewer to conduct the survey with 50 male and 50 female cinema-goers as they leave the cinema.

A disadvantage of this method is the possibility of bias in the selection process, as interviewers may tend to choose the easiest way to fulfill the survey quota eg. selecting those who are more open and the easiest to approach; interviewing couples (with 1 female and 1 male) who may tend to give the same opinion.

6. **Topic: Correlation Coefficient and Linear Regression**

(i)



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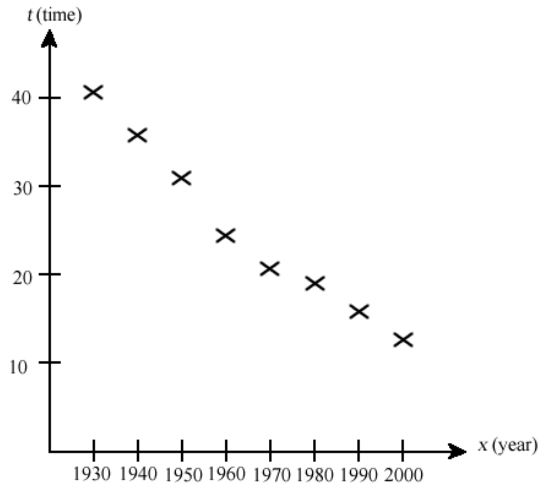
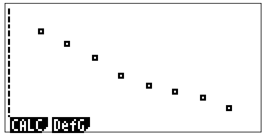




(ii) As far as the data in the scatter diagram is concerned, the linear model is appropriate since its calculated value of $r = -0.986$ indicates a strong negative linear correlation.

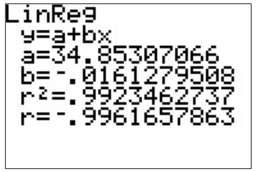
In the context of the question, however, it is unlikely that the world record will decrease linearly with time since its likely to be increasingly difficult to break it as we approach the limits of our human abilities as time goes by. Hence a non-linear model with a negative exponential function may be more appropriate than a linear model.

(iii) A quadratic model (with a minimum point) would not be appropriate since the world record time can only decrease or remain the same as the years go by. Hence there cannot be a portion where t increases as x increases in the long term.

(iv) By generating another list $\Rightarrow$ L = ln t and using G.C., we have the line of regression:

$$\Rightarrow \quad \ln t = 34.853 - 0.016127x$$
$$\approx 34.9 - 0.0161x$$

Coefficient of correlation, $r = -0.99616$



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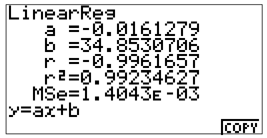

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Note: world record time = 3min 30sec + t sec

Sub $x = 2010$, $\therefore t = 11.447$

$$\approx 11.4 \text{ (3 sig. fig.)}$$

$\therefore$ **World record time as of 1st January 2010 is 3 minutes 41.4 seconds.**

As the 2010 world record time is predicted through extrapolating our data well beyond the year 2000, it is not reliable despite its strong correlation.

7. **Topic: Probability, Differentiation**

(i) Given $p = 25$,
Probability that a randomly chosen component is faulty
= P(component supplied by A is faulty or component supplied by B is faulty)
= P(component supplied by A is faulty) + P(component supplied by B is faulty)
$= \frac{25}{100} \times 0.05 + \frac{75}{100} \times 0.03$
= **0.035**

(ii) For a general value of p,

$$f(p) = \frac{\frac{p}{100}\times 0.05}{\frac{p}{100}\times 0.05 + \frac{100-p}{100}\times 0.03}$$

P(component supplied by A is faulty) / P(randomly chosen component is faulty)

$$= \frac{\frac{p}{100}\times 0.05}{\frac{1}{100}[0.05p + 3 - 0.03p]}$$

$$= \frac{0.05p}{0.02p+3}$$ **(Shown)**

$$f'(p) = \frac{(0.02p+3)0.05 - 0.05p(0.02)}{(0.02p+3)^2}$$

$$= \frac{0.15}{(0.02p+3)^2}$$

for $0 \le p \le 100$, $(0.02p+3)^2 > 0$

$\therefore f'(p) = \frac{0.15}{(0.02p+3)^2} > 0$

$\therefore$ f is an increasing function for $0 \le p \le 100$. **(Proved)**

The increasing function f(p) shows that as the company buys a greater percentage of its electric components from supplier A, the probability of a faulty component that is randomly packed from supplier A increases. This translates into a greater likelihood of receiving a greater number of faulty components from supplier A.

8. **Topic: Permutations and Combinations**

ELEVATED
E – 3, L – 1, V – 1, A – 1, T – 1, D – 1

(i) No. of ways to be arranged (without restrictions) $= \frac{8!}{3!}$ = **6720**

Treat D and T as one group

(ii) No. of ways for T and D next to each other $= \frac{7!}{3!} \times 2! = 1680$
$\therefore$ No. of ways of T and D must not be next to each other = 6720 – 1680 = **5040**

(iii) No. of ways for consonants (L, V, T, D) and vowels must be alternate
$= 4! \times \frac{4!}{3!} \times 2$
= **192**

C V C V C V C V
or V C V C V C V C

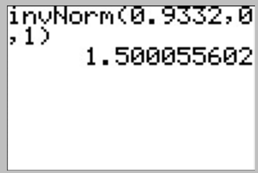
(iv) Case 1: X E X X E X X E X = 5! × 2! = 240
Case2: E X X X E X X = 5! × 2! = 240
Note: X denotes any letter that is not E.
$\therefore$ No. of ways between any two Es must be at least 2 other letters
= 240 + 240
= **480**

9. **Topic: Normal Distributions**

Let M be the random variable of the thickness in cm of a mechanics textbook.

(i)

$$M \sim N(2.5, 0.1^2)$$
$$\therefore \bar{M} \sim N\left(2.5, \frac{0.1^2}{n}\right)$$
$$\text{Given } P(\bar{M} > 2.53) = 0.0668$$
$$\text{Then } P\left(Z > \frac{2.53-2.5}{\frac{0.1}{\sqrt{n}}}\right) = 0.0668$$
$$1 - P\left(Z < \frac{2.53-2.5}{\frac{0.1}{\sqrt{n}}}\right) = 0.0668$$
$$P(Z < 0.3\sqrt{n}) = 0.9332$$
$$0.3\sqrt{n} = 1.5$$
$$\boldsymbol{n \approx 25}$$



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(ii) Let S be the random variable of the thickness in cm of a statistics textbook.

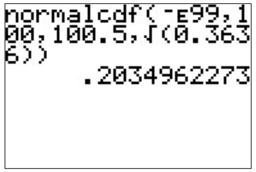
$$S \sim N(2.0, 0.08^2)$$
$$\text{Let } M_T = M_1 + M_2 + M_3 + \ldots + M_{21}$$
$$\therefore M_T \sim N(21 \times 2.5, 21 \times 0.1^2)$$
$$= N(52.5, 0.21)$$
$$\text{Let } S_T = S_1 + S_2 + S_3 + \ldots + S_{24}$$
$$\therefore S_T \sim N(24 \times 2, 24 \times 0.08^2)$$
$$= N(48.0, 0.1536)$$
$$\Rightarrow M_T + S_T \sim N(100.5, 0.3636)$$

P(21 mechanic textbooks & 24 statistics textbooks will fit into a bookshelf of length 1m)

$= P(M_T + S_T \le 100 \text{ cm})$

$= 0.20349$

$\approx$ **0.203 (3 sig. fig.)**



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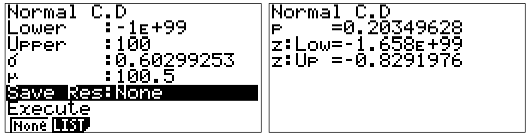

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(iii)

$$\text{Let } D = S_1 + S_2 + S_3 + S_4 - 3M$$
$$\text{Then } E(D) = 4E(S) - 3E(M)$$
$$= 4(2) - 3(25)$$
$$= 0.5$$
$$\text{and } Var(D) = 4Var(S) + 9Var(M)$$
$$= 4(0.08)^2 + 9(0.01)^2$$
$$= 0.1156$$

$\therefore D \sim N(0.5, 0.1156)$

P(The total thickness of 4 statistics textbooks < 3 times the thickness of 1 mechanics textbook)

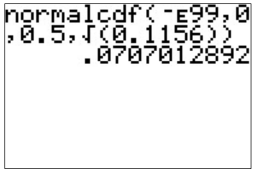
$= P(S_1 + S_2 + S_3 + S_4 < 3M)$

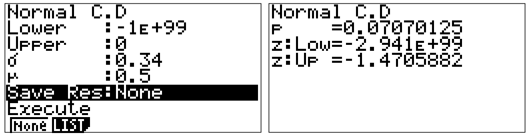
$= P(S_1 + S_2 + S_3 + S_4 - 3M < 0)$

$= P(D < 0)$

$= 0.07070$

$\approx$ **0.0707 (3 sig. fig.)**





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(iv) **The thickness of a mechanics textbook is independent of the thickness of a statistics textbook.**

> For normal distributions, variables are assumed to be independent of each other.

10. **Topic: Hypothesis Testing**

(i) Unbiased estimate of the mean, $\bar{x} = \frac{\sum x}{n}$

$= \frac{86.4}{8}$

$= \mathbf{9.6}$

Unbiased estimate of the variance of $X = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - (\bar{x})^2\right]$

$s^2 = \frac{9}{8}\left[\frac{835.02}{9} - (9.6)^2\right]$

$= \mathbf{0.81}$

(ii) Assumption: Mass of sugar follows a normal distribution.

$H_o: \mu = 10$ grams

$H_1: \mu \neq 10$ grams

$T = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$

$= \frac{9.6-10}{\sqrt{\frac{0.81}{9}}}$

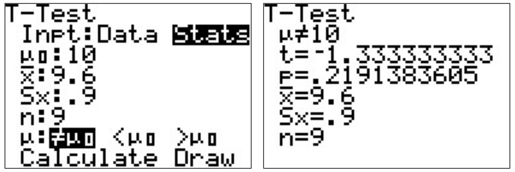
$= -1.333$ where $\mu = 10$, $s = \sqrt{0.81}$ and $n = 9$

> Refer to table in MF15 to find critical value for the t-distribution.

By G.C., p-value = 0.2191($>$ 0.05)

Hence we do not reject H_o and conclude that at the 5% level of significance, there is insufficient evidence to conclude that the mass of the packet is not 10 grams.

In this case, the sample size is small (say, < 30). It's not large enough to assume a normal distribution according to central limit theorem.



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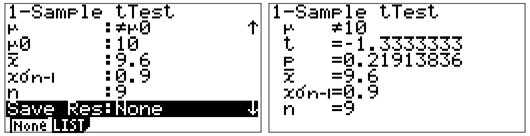

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(iii) As the population variance of X is known, the z-test is carried out instead of the t-test. i.e. $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$\therefore z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$

11. **Topic: Binomial, Poisson and Normal Distributions**

(i) The assumptions needed for R to be well modeled by a binomial distribution:

Given $R \sim B(n, p)$,

(a) The colour of the car is either red or not red.

(b) The trials are independent i.e. the colour of the car in each observation is independent of the colour of the car in every other observation.

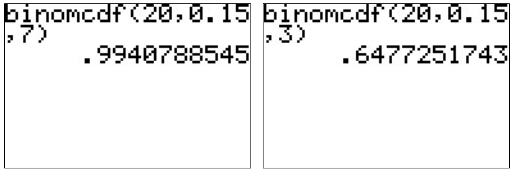
(ii) Given also that $n = 20$, $p = 0.15$

$P(4 \le R < 8) = P(4 \le R \le 7)$

$= P(R \le 7) - P(R \le 3)$

$= 0.99407 - 0.64772$

$\approx$ **0.346 (3 sig. fig.)**



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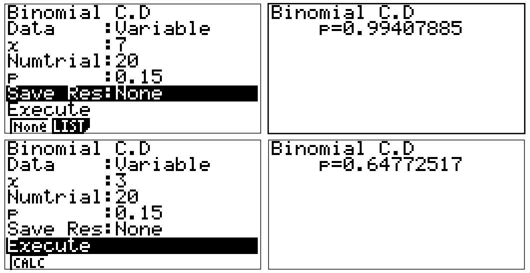

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(iii) Given that $n = 240$, $p = 0.3$

Since $n = 240$ (> 50) is large, $np = 72 > 5$ and $n(1-p) = 168 > 5$

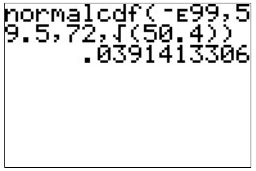
the binomial distribution can be approximated using the normal distribution with mean $np = 72$ and variance $np(1-p) = 50.4$

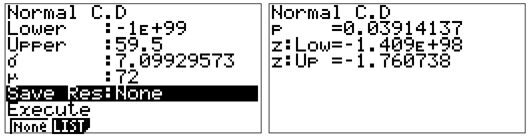
$\therefore R \sim N(72, 50.4)$ approximately.

$P(R < 60) \approx P(R < 59.5)$ [Continuity Correction]

$=$ **0.0391 (3 sig. fig.)**

Use continuity correction to approximate a discrete distribution (i.e binomial) by a continuous distribution (i.e normal).





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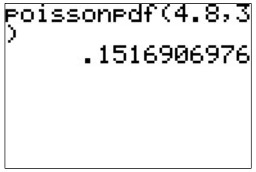
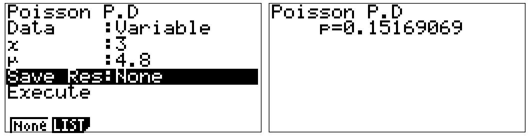
(iv) Given that $n = 240$ and $p = 0.02$

Since $n > 50$, $p < 0.1$ and $np = 4.8 < 5$, the binomial distribution can be approximated using the poisson districution with mean $\lambda = np = 4.8$

i.e. $R \sim Po(4.8)$ approximately

$P(R = 3) \approx$ **0.1517 (4 d.p.)**





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(v) Given that $n = 20$ and $P(R = 0 \text{ or } 1) = 0.2$

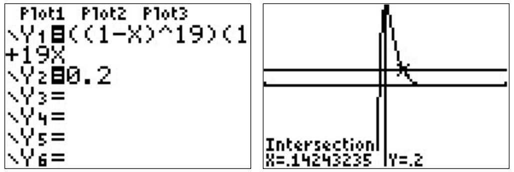
Then $\quad P(R = 0) + P(R = 1) = 0.2$

$${}^nC_r p^r q^{n-r}$$

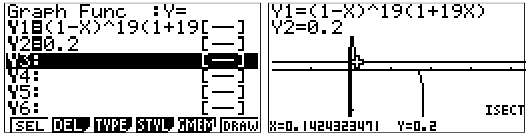
$$(1-p)^{20} + \binom{20}{1} p^1 (1-p)^{19} = 0.2$$

$$(1-p)^{19}(1+19p) = 0.2$$

Using G.C., $\quad$ $\boldsymbol{p} = $ **0.142 (3 sig. fig.)**



TI-84 Plus



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