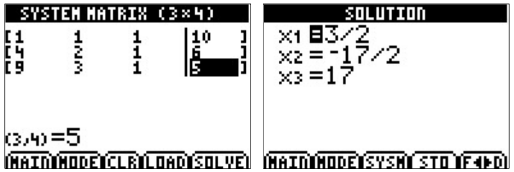
MATHEMATICS (H2) 9740/01

Paper 1 Suggested Solutions — **October/November 2009**

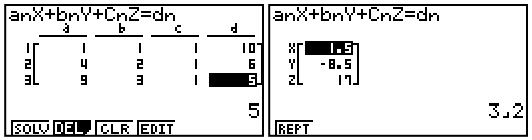
1. **Topic: Matrices**

(i) Given that u_n is a quadratic polynomial in n,
Let $u_n = an^2 + bn + c$, where a, b and c are constants

When $u_1 = 10$, $u_1 = a + b + c = 10$
When $u_2 = 6$, $u_2 = 4a + 2b + c = 6$
When $u_3 = 5$, $u_3 = 9a + 3b + c = 5$



TI-84 Plus



Casio fx-9860G

Using G.C., $a = \frac{3}{2}$, $b = -\frac{17}{2}$ and $c = 17$

$\therefore \boldsymbol{u_n = \frac{3}{2}n^2 - \frac{17}{2}n + 17}$

(ii) $\frac{3}{2}n^2 - \frac{17}{2}n + 17 > 100$

$3n^2 - 17n + 34 > 200$

$3n^2 - 17n - 166 > 0$ ………………… (1)

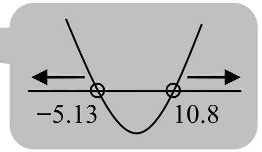
Consider: $3n^2 - 17n - 166 = 0$

$$n = \frac{17 \pm \sqrt{17^2 + 4(166)(3}}{2(3)}$$

$$\therefore n = \frac{17 - \sqrt{2281}}{6} \text{ or } \frac{17 + \sqrt{2281}}{6}$$

$$\approx -5.13 \text{ or } 10.8$$

From (1), $n < -5.13$ or $n > 10.8$



Since $n \in \mathbb{Z}^+$, $\boldsymbol{n \geq 11}$ is the set of values of n.

2. **Topic: Definite Integrals**

By Partial Fractions:

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)}$$

$$= \frac{A}{2-x} + \frac{B}{2+x}$$

$$\Rightarrow 1 = A(2+x) + B(2-x)$$

$x = -2$, $B = \frac{1}{4}$

$x = 2$, $A = \frac{1}{4}$

$$\int_0^1 \frac{1}{4-x^2}\,dx = \int_0^1 \left[\frac{1}{4(2-x)} + \frac{1}{4(2+x)}\right] dx$$

$$= \frac{1}{4}\left[\ln\left|\frac{2+x}{2-x}\right|\right]_0^1$$

$$= \frac{1}{4}\ln 3$$

Alternatively, quote formula from formula list: $\left[\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)\right]$

$$\int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1-p^2x^2}}\,dx = \frac{1}{p}\int_0^{\frac{1}{2p}} \frac{p}{\sqrt{1-(px)^2}}\,dx$$

$$\frac{d}{dx}[\sin^{-1} f(x)] = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$$

$$= \frac{1}{p}[\sin^{-1}(px)]_0^{\frac{1}{2p}}$$

$$= \frac{1}{p}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$$

$$= \frac{\pi}{6p}$$

Given $\int_0^1 \frac{1}{4-x^2}\,dx = \int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1-p^2x^2}}\,dx$,

From above, $\frac{1}{4}\ln 3 = \frac{\pi}{6p}$

$$\therefore p = \frac{4\pi}{6\ln 3} = \mathbf{\frac{2\pi}{3\ln 3}}$$

3. **Topic: $\sum$ Notation**

(i) $$\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} = \frac{n(n+1)-2(n+1)(n-1)+n(n-1)}{n(n-1)(n+1)}$$

$$= \frac{n^2+n-2n^2+2+n^2-n}{n(n^2-1)}$$

$$= \frac{2}{n^3-n}$$

$$\therefore A = \mathbf{2}$$

(ii) Using 3(i) and Method of Difference:

$$\sum_{r=2}^{n} \frac{1}{r^3-r} = \frac{1}{2}\sum_{r=2}^{n} \frac{2}{r^3-r}$$

$$= \frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}\right)$$

$$= \frac{1}{2}\Bigg[\frac{1}{1} - \frac{2}{2} + \frac{1}{3}$$
$$+ \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$
$$+ \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$$
$$+ \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$$
$$\vdots$$
$$\vdots$$
$$\vdots$$
$$+ \frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-1}$$
$$+ \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}$$
$$+ \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\Bigg]$$

$$= \frac{1}{2}\left[1 - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1}\right]$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right)$$

$$\therefore \sum_{r=2}^{n} \frac{1}{r^3-r} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right)$$

$$\lim_{n\to\infty} \sum_{r=2}^{n} \frac{1}{r^3-r} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right)$$

$$\sum_{r=2}^{\infty} \frac{1}{r^3-r} = \frac{1}{2}\left(\frac{1}{2} - 0 + 0\right) = \frac{1}{4}$$

(iii) **$\therefore$ the series converges to a constant value of $\frac{1}{4}$ as $n \to \infty$.**

4. **Topics: Functions, Graphs, Integration**

Given $f(x) = \begin{cases} 7 - x^2 \text{ for } 0 < x \le 2 \\ 2x - 1 \text{ for } 2 < x \le 4 \end{cases}$ and $f(x) = f(x + 4)$

> Graph is a periodic curve that repeats itself at every 4-unit interval.

(i)
$$\begin{aligned} f(27) &= f(23+4) \\ &= f(23) \\ &= f(19+4) \\ &= f(19) \\ &= f(15) \\ &= f(11) \\ &= f(7) \\ &= f(3) \\ &= 2(3) - 1 \\ &= \mathbf{5} \end{aligned}$$

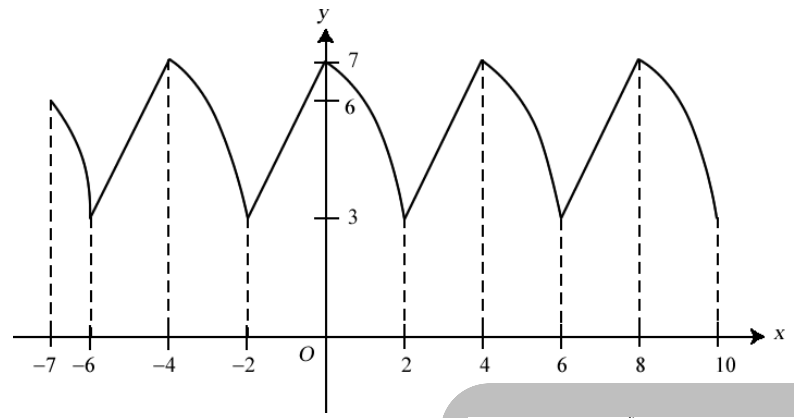
> $x = 3$, lies in interval $2 < x \le 4$
> $\Rightarrow f(x) = 2x - 1$

$$\begin{aligned} f(45) &= f(41+4) \\ &= f(41) \\ &= f(37) \\ &= f(33) \\ &= \ldots \\ &= f(1) \\ &= 7 - 1 \\ &= \mathbf{6} \end{aligned}$$

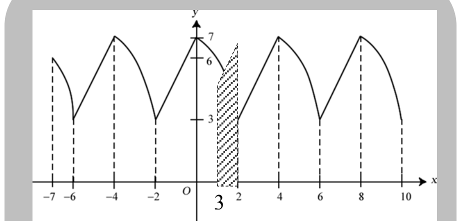
> $x = 1$, lies in interval $0 < x \le 2$
> $\Rightarrow f(x) = 7 - x^2$

$$\therefore f(27) + f(45) = 5 + 6 = \mathbf{11}$$

(ii)



$$\begin{aligned} f(-7) &= f(-7+4) \\ &= f(-3) \\ &= f(-3+4) \\ &= f(+1) \\ &= 7 - 1^2 \\ &= 6 \end{aligned}$$



Area of trapezium to be subtracted

(iii)
$$\begin{aligned} \int_{-4}^{3} f(x)\, dx &= \int_{-4}^{4} f(x)dx - \int_{3}^{4} f(x)dx \\ &= 2\int_{0}^{4} f(x)dx - \text{area of trapezium} \\ &= 2\left[\int_0^2 (7 - x^2)dx + \int_2^4 2x - 1dx\right] - \tfrac{1}{2}(5+7)(1) \\ &= 2\left\{\left[7x - \tfrac{1}{3}x^3\right]_0^2 + \tfrac{1}{2}(3+7)(2)\right\} - 6 \\ &= 2\left\{\left(14 - \tfrac{8}{3}\right) + 10\right\} - 6 \\ &= \mathbf{36\tfrac{2}{3}\ unit^2} \end{aligned}$$

5. **Topic: Mathematical Induction**

Let P_n be the statement such that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ where $n \geq 1$

When $n = 1$,

$$\begin{aligned}
\text{L.H.S.} &= \sum_{r=1}^{1} r^2 \\
&= 1^2 \\
&= 1 \\
\text{R.H.S.} &= \frac{1}{6}(1)(1+1)(2+1) \\
&= \frac{1}{6}(2)(3) \\
&= 1
\end{aligned}$$

$\therefore$ L.H.S. = R.H.S.

$\therefore P_1$ is true

Assume P_k is true i.e. $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$, for some $k \in \mathbb{Z}^+$

To show that P_{k+1} is also true i.e. $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)(k+2)[2(k+1)+1]$,

$$\begin{aligned}
\text{L.H.S.} &= \sum_{r=1}^{k+1} r^2 \\
&= \sum_{r=1}^{k} r^2 + \overbrace{(k+1)^2}^{(k+1)^{\text{th}}\text{ term}} \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\
&= \frac{1}{6}(k+1)[2k^2 + 7k + 6] \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)(k+2)[2(k+1)+1] \\
&= \text{R. H. S.}
\end{aligned}$$

Bring out $\frac{1}{6}(k+1)$ factor since it is found on RHS.

i.e. P_{k+1} is true if P_k is true.

Since P_k is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

$$\begin{aligned}
\sum_{r=n+1}^{2n} r^2 &= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2 \\
&= \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}n(n+1)(2n+1) \\
&= \frac{1}{6}n(2n+1)[8n+2-n-1] \\
&= \mathbf{\frac{1}{6}n(2n+1)(7n+1)}
\end{aligned}$$

Using given equation: $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

6. **Topic: Graphing Techniques**

Given C_1 : $y = \frac{x-2}{x+2}$ and C_2: $\frac{x^2}{6} + \frac{y^2}{3} = 1$

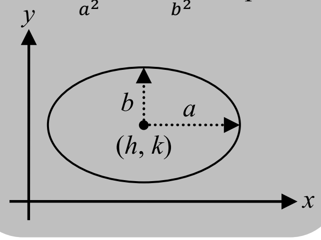
$y = 1 - \frac{4}{x+2}$

$\frac{x^2}{(\sqrt{6})^2} + \frac{y^2}{(\sqrt{3})^2} = 1$

C_2 is an ellipse

$y = \frac{a}{x-h} + k$
Vertical asymptote: $x = h$
Horizontal asymptote: $y = k$

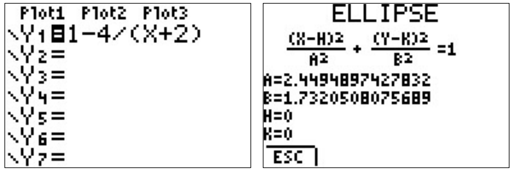
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$



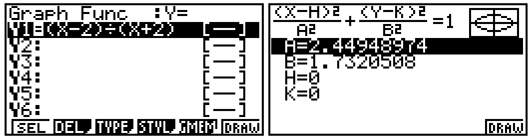
(i) Vertical asymptotes: $x = -2$
Horizontal asymptotes: $y = 1$

C_1: When $x = 0$, $y = -1$
When $y = 0$, $x = 2$

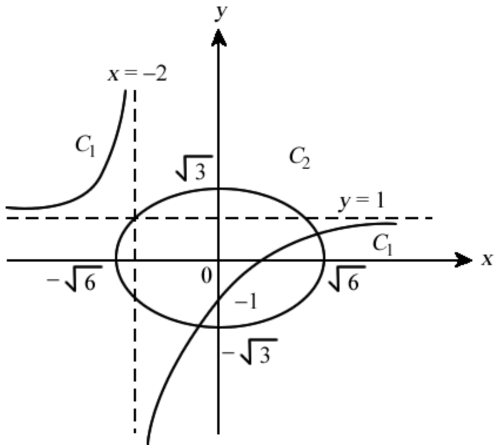
C_2: When $x = 0$, $y = \pm\sqrt{3}$
When $y = 0$, $x = \pm\sqrt{6}$

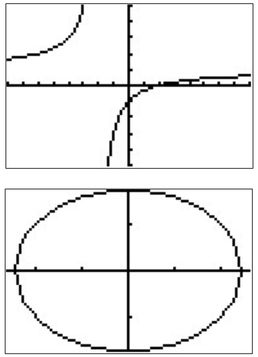


TI-84 Plus



Casio fx-9860G





(ii) Given C_1 : $y = \frac{x-2}{x+2}$ ……….. (1)

C_2: $\frac{x^2}{6} + \frac{y^2}{3} = 1$

$x^2 + 2y^2 = 6$…………..... (2)

Sub (1) into (2),

$$x^2 + 2\left(\frac{x-2}{x+2}\right)^2 = 6$$

$$x^2(x+2)^2 + 2(x-2)^2 = 6(x+2)^2$$

$$2(x-2)^2 = 6(x+2)^2 - x^2(x+2)^2$$

$$2(x-2)^2 = (6-x^2)(x+2)^2 \text{ (Shown)}$$

(iii) Using G.C.:

$x \approx -0.515$ or $x \approx 2.45$ (3 sig. fig.)

7. **Topic: Maclaurin's Series**

(i) Given $f(x) = e^{\cos x}$ $\Rightarrow f(0) = e$

Let $y = e^{\cos x}$

$$\frac{dy}{dx} = -\sin x\, e^{\cos x} \quad \Rightarrow f'(0) = 0$$

$$\frac{dy}{dx} = -y\sin x$$

$$\frac{d^2y}{dx^2} = -\sin x\frac{dy}{dx} - y\cos x \quad \Rightarrow f''(0) = -e$$

By Maclaurin's theorem:

$$f(x) = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2$$

$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!} + \ldots$

$$= e + 0x + \frac{-e}{2}x^2$$

$$= \mathbf{e - \frac{1}{2}ex^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

(ii) $$\frac{1}{a+bx^2} = (a+bx^2)^{-1}$$

$$= a^{-1}\left(1 + \frac{b}{a}x^2\right)^{-1}$$

Factorise a^{-1} to make into general term $(1 + ax)^n$

$(1 + ax)^n = 1 + nax + \frac{n(n-1)}{2!}(ax)^2 + \ldots$

$$= \frac{1}{a}\left[1 + \frac{(-1)}{1!}\left(\frac{b}{a}x^2\right) + ..\right]$$

$$= \frac{1}{a}\left[1 - \frac{b}{a}x^2 + ..\right]$$

$$= \frac{1}{a} - \frac{b}{a^2}x^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Given 1st two non-zero terms of (1) = 1st two non-zero terms of (2),

$$\therefore e - \frac{1}{2}ex^2 = \frac{1}{a} - \frac{b}{a^2}x^2$$

Comparing constant terms:

$$e = \frac{1}{a}$$

$$\boldsymbol{a = \frac{1}{e}}$$

Comparing coefficients of x^2:

$$-\frac{1}{2}e = -\frac{b}{a^2}$$

$$b = \frac{1}{2}ea^2$$

$$= \frac{1}{2}e\left(\frac{1}{e^2}\right)$$

$$= \mathbf{\frac{1}{2e}}$$

8. **Topic: Geometric Progressions**

(i) Given instrument A, the bars form a G.P. series.

Given $T_1 = a = 20$ cm

Given $T_{25} = ar^{24} = 5$ cm

$$r^{24} = \frac{1}{4}$$

$$r = [2^{-2}]^{\frac{1}{24}}$$

$$= 2^{-\frac{1}{12}}$$

$$\therefore \text{Total length of } n \text{ bars} = \frac{a(1-r^n)}{1-r}$$

$$= \frac{20\left[1-\left(2^{-\frac{1}{12}}\right)^n\right]}{1-2^{-\frac{1}{12}}}$$

Since $r = 2^{-\frac{1}{12}} < 1$, $\left(2^{-\frac{1}{12}}\right)^n \to 0$ as $n \to \infty$

$$\therefore \text{Total length} = \frac{20}{1-2^{-\frac{1}{12}}}$$

$$\approx 356.34 < 357 \text{ cm } \textbf{(Shown)}$$

(ii) Given instrument B consists of only 25 bars which are identical to the 1st 25 bars of instrument A.

Total length of 25 bars instrument B, $L = \dfrac{20\left[1-\left(2^{-\frac{1}{12}}\right)^{25}\right]}{1-2^{-\frac{1}{12}}}$

$\therefore L \approx$ **272 cm (3 sig. fig.)**

$$\begin{aligned}\text{Length of 13}^{\text{th}}\text{ bar} &= T_{13}\\ &= 20\left(2^{-\frac{1}{12}}\right)^{13-1}\\ &= 20\left(2^{-\frac{1}{12}}\right)^{12}\\ &= \mathbf{10\ cm}\end{aligned}$$

$T_n = ar^{n-1}$

(iii) Using results from 8(ii),

$$\begin{aligned}L &= 272\text{ cm}\\ \frac{25}{2}(a+l) &= 272 \ \ldots\ldots\ldots\ldots (1)\end{aligned}$$

$\frac{n}{2}(a+l)$ where a is value of 1st term and l is value of last term

Given $T_{25} = 5$ cm,

$$l = 5 \ \ldots\ldots\ldots\ldots (2)$$

Sub (2) into (1), $\frac{25}{2}(a+5) = 272$

$$a = 16.76\text{ cm}$$

From (2), we also have:

$$\begin{aligned}a + 24d &= 5\\ 16.76 + 24d &= 5\\ d &= -0.49\text{ cm}\\ \therefore \text{Value of } d &= \mathbf{-0.49\ cm}\\ \text{Longest bar} &= a\\ &= \mathbf{16.76\ cm\ (exact)}\end{aligned}$$

length of 25th bar

1st bar will be longest because d is negative, hence subsequent bars will get shorter

9. **Topic: Complex Numbers**

(i) Given $z^7 - (1+i) = 0$

$$\begin{aligned}z^7 &= (1+i)\\ z^7 &= 2^{\frac{1}{2}}e^{\left(2k+\frac{1}{4}\right)\pi i}\\ z &= 2^{\frac{1}{14}}e^{\frac{1}{7}\left(2k+\frac{1}{4}\right)\pi i}, \text{ where } k = -3, -2, -1, 0, 1, 2, 3\end{aligned}$$

$z^n = re^{i(\theta+2k\pi)}$

$z = r^{\frac{1}{n}}e^{\frac{1}{n}(\theta+2k\pi)i}$

Hence for $k = -3$, $z = 2^{\frac{1}{14}}e^{-\frac{23}{28}\pi i} = z_7$

for $k = -2$, $z = 2^{\frac{1}{14}}e^{-\frac{15}{28}\pi i} = z_6$

for $k = -1$, $z = 2^{\frac{1}{14}}e^{-\frac{7}{28}\pi i} = z_5$

for $k = 0$, $z = 2^{\frac{1}{14}}e^{\frac{1}{28}\pi i} = z_1$

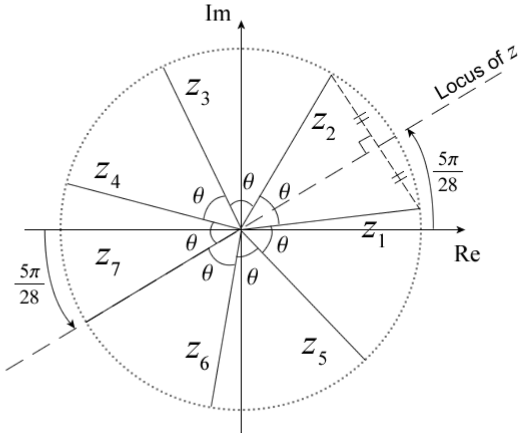
for $k = 1$, $z = 2^{\frac{1}{14}}e^{\frac{9}{28}\pi i} = z_2$

for $k = 2$, $z = 2^{\frac{1}{14}}e^{\frac{17}{28}\pi i} = z_3$

for $k = 3$, $z = 2^{\frac{1}{14}}e^{\frac{25}{28}\pi i} = z_4$

$\therefore$ **The roots are** $2^{\frac{1}{14}}e^{-\frac{23}{28}\pi i}$, $2^{\frac{1}{14}}e^{-\frac{15}{28}\pi i}$, $2^{\frac{1}{14}}e^{-\frac{7}{28}\pi i}$, $2^{\frac{1}{14}}e^{\frac{1}{28}\pi i}$, $2^{\frac{1}{14}}e^{\frac{9}{28}\pi i}$, $2^{\frac{1}{14}}e^{\frac{17}{28}\pi i}$ **and** $2^{\frac{1}{14}}e^{\frac{25}{28}\pi i}$

(ii)



Note $|z_1| = |z_2| = |z_3| = |z_4| = |z_5| = |z_6| = |z_7| = 2^{\frac{1}{14}}$

and $\theta = \frac{8}{28}\pi = \frac{2}{7}\pi$

(iii) Given $|z - z_1| = |z - z_2|$ ………………… (1)

for $z = 0 + 0\text{i}$, we have $|0 - z_1| = |z_1| = 2^{\frac{1}{14}}$

for $z = 0 + 0\text{i}$, we have $|0 - z_2| = |z_2| = 2^{\frac{1}{14}}$

∴ (0, 0) is one of the locus points for (1)

Hence, the locus of all points z s.t. $|z - z_1| = |z - z_2|$ passes through origin.

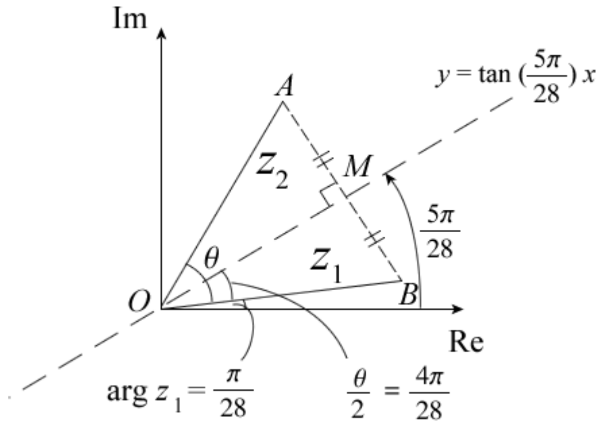
Since $OA = OB$, OM intersects ∠ AOB

arg $(z_1) + \frac{\theta}{2}$

$$\therefore \text{Gradient } OM = \tan\left(\frac{\pi}{28} + \frac{4\pi}{28}\right) = \tan\left(\frac{5\pi}{28}\right)$$

∴ Cartesian equation of OM: $\boldsymbol{y = \tan\left(\frac{5\pi}{28}\right).\, x}$

$y - 0 = \left[\tan\left(\frac{5\pi}{28}\right)\right](x - 0)$



10. **Topic: Vectors**

p_1: $\mathbf{r}.\begin{pmatrix}2\\1\\3\end{pmatrix} = 1$ Let $\mathbf{n}_1 = \begin{pmatrix}2\\1\\3\end{pmatrix}$ $|\mathbf{n}_1| = \sqrt{14}$

p_2: $\mathbf{r}.\begin{pmatrix}-1\\2\\1\end{pmatrix} = 2$ Let $\mathbf{n}_2 = \begin{pmatrix}-1\\2\\1\end{pmatrix}$ $|\mathbf{n}_2| = \sqrt{6}$

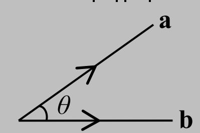
(i) Let θ be the acute angle between p_1 and p_2

$$\text{Then } \cos\theta = \left|\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1|.|\mathbf{n}_2|}\right| = \left|\frac{\begin{pmatrix}2\\1\\3\end{pmatrix}.\begin{pmatrix}-1\\2\\1\end{pmatrix}}{\sqrt{14}\sqrt{6}}\right|$$

Modulus because angle is acute

$$\therefore \theta = \cos^{-1}\left(\frac{3}{\sqrt{84}}\right) = 70.89° \approx \mathbf{70.9° \text{ (1 d.p.)}}$$

Scalar product of two vectors **a** and **b**:
a · b = |a||b| cos θ



(ii) $$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix}2\\1\\3\end{pmatrix} \times \begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}-5\\-5\\5\end{pmatrix} = -5\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}$$

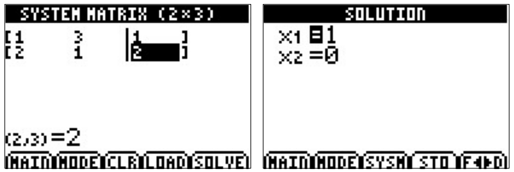
Let **d** be the direction vector of l.

$\mathbf{d} \,//\, \mathbf{n}_1 \times \mathbf{n}_2$, ∴ Take $\mathbf{d} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

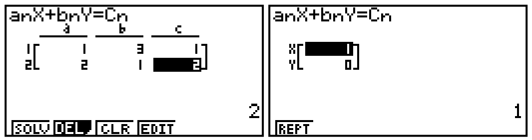
p_1: $\mathbf{r}.\begin{pmatrix}2\\1\\3\end{pmatrix} = 1 \quad \Rightarrow 2x + y + 3z = 1$

p_2: $\mathbf{r}.\begin{pmatrix}-1\\2\\1\end{pmatrix} = 2 \quad \Rightarrow -x + 2y + z = 2$

Solve by simultaneous equations by letting $x = 0$.



TI-84 Plus



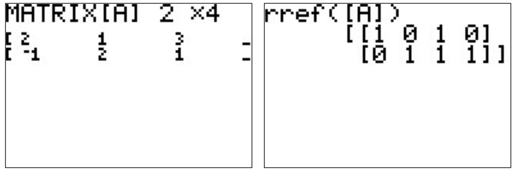
Casio fx-9860G

Using G.C., the point of intersection of p_1 and p_2 is $\begin{pmatrix}0\\1\\0\end{pmatrix}$

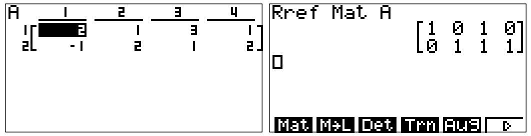
∴ Vector equation of line l: $\mathbf{r} = \begin{pmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{pmatrix} + \boldsymbol{\lambda}\begin{pmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{-1}\end{pmatrix}, \boldsymbol{\lambda} \in \mathbb{R}$ .... (1)

ALTERNATE APPROACH

Using rref,



TI-84 Plus



Casio fx-9860G

$$x + z = 0 \Rightarrow x = -z$$
$$y + z = 1 \Rightarrow y = 1 - z$$

Vector equation of l:
$$\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-z\\1-z\\z\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$$ where λ is a parameter

(iii) Given p_3: $2x + y + 3z - 1 + k(-x + 2y + z - 2) = 0$

$$(2-k)x + (2k+1)y + (k+3)z = 2k+1$$

Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\quad p_3$: $\mathbf{r}.\begin{pmatrix} 2-k \\ 2k+1 \\ k+3 \end{pmatrix} = 2k+1$ …………………… (2)

Let $\mathbf{n_3} = \begin{pmatrix} 2-k \\ 2k+1 \\ k+3 \end{pmatrix}$

From (1), we know $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ lies on line l and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.\begin{pmatrix} 2-k \\ 2k+1 \\ k+3 \end{pmatrix} = 2k+1$

$\therefore \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ also lies on p_3.

$$\begin{aligned} \mathbf{d}\cdot\mathbf{n_3} &= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 2-k \\ 2k+1 \\ k+3 \end{pmatrix} \\ &= 2 - k + 2k + 1 - (k+3) \\ &= 0 \end{aligned}$$

Two non-zero vectors **a** and **b** are $\perp$ if $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 90° = 0$

$\mathbf{d} \perp \mathbf{n_3} \Rightarrow \mathbf{d}$ // plane p_3

Since $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ lies in p_3 and $\mathbf{d}$ // p_3, l lies in p_3 for any k.

ALTERNATE APPROACH

$2x + y + 3z - 1 + k(-x + 2y + z - 2) = 0 \qquad - (1)$

sub $x = -z$ and $y = 1 - z$ into L.H.S. of (1),

$$\begin{aligned} \text{L.H.S.} &= 2(-z) + (1-z) + 3z - 1 + k[z + 2(1-z) + z - 2] \\ &= 0 \\ &= \text{R.H.S.} \end{aligned}$$

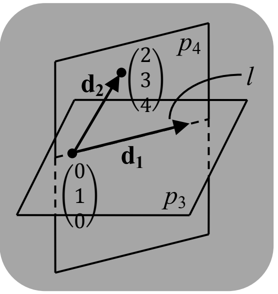
$\therefore l$ lies in p_3 and is independent of k.

Given $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ lies on p_3 then $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.\begin{pmatrix} 2-k \\ 2k+1 \\ k+3 \end{pmatrix} = 2k+1 \quad \Rightarrow k = -3$

From (2), $\therefore p_3$: $\mathbf{r}\cdot\begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix} = -5$

$$\Rightarrow \quad 5x - 5y = -5$$

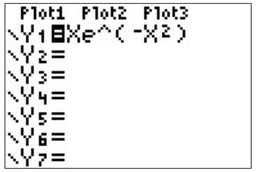
$$\boldsymbol{x - y = -1, z \in \mathbb{R}}$$



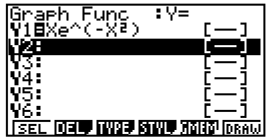
11. **Topic: Graphs, Integration by Substitution, Applications of Integration (Volume)**

Given $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = xe^{-x^2}$

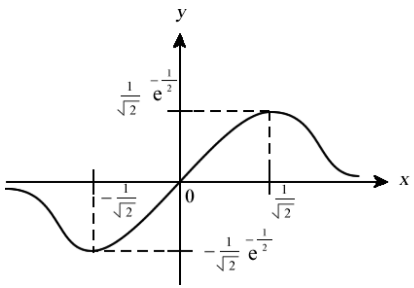
(i)

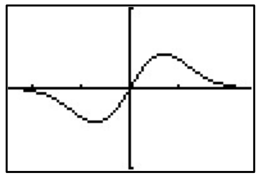


TI-84 Plus



Casio fx-9860G





(ii) $y = xe^{-x^2}$

$\frac{dy}{dx} = e^{-x^2} + xe^{-x^2}(-2x)$

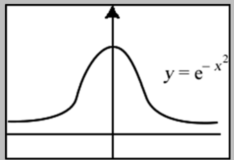
$= e^{-x^2}[1 - 2x^2]$

When $\frac{dy}{dx} = 0$, $e^{-x^2}[1 - 2x^2] = 0$

Since $e^{-x^2} > 0$, $1 - 2x^2 = 0$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$



From above graph, e^{-x^2} is always +ve

When $x = -\frac{1}{\sqrt{2}}$, $y = -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$

When $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$

∴ **The coordinates of the turning points are $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}\right)$.**

(iii) Use $u = x^2$, $du = 2x\,dx$
When $x = 0$, $u = 0$ and when $x = n$, $u = n^2$

$\int_0^n f(x)\,dx = \int_0^n xe^{-x^2}\,dx$

$= \frac{1}{2}\int_0^n 2xe^{-x^2}\,dx$

$= \frac{1}{2}\int_0^{n^2} e^{-u}\,du$

$= \frac{1}{2}[-e^{-u}]_0^{n^2}$

$= \frac{1}{2}[-e^{-n^2} + 1]$

$= \frac{1}{2}(1 - e^{-n^2})$

∴ the area of the region between the curve and the positive x-axis

$= \lim_{n\to\infty}\int_0^n f(x)\,dx$

$\approx \frac{1}{2}(1 - 0)$ ($e^{-n^2} \to 0$ as $n\to\infty$)

$=$ **$\frac{1}{2}$ unit²**

(iv) Using (i):

$\int_{-2}^{2}|f(x)|\,dx = 2\int_0^2 f(x)\,dx$

$= 2\left[\frac{1}{2}(1 - e^{-2^2})\right]$

$=$ **$1 - e^{-4}$**

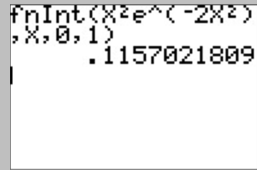
(v) Volume about x-axis $= \pi\int_0^1 y^2\,dx$

$= \pi\int_0^1 x^2e^{-2x^2}\,dx$

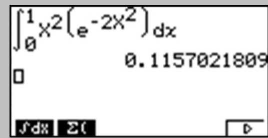
$= \pi(0.11570218)$

$= 0.36348$

$\approx$ **0.363 unit³ (3 sig. fig.)**



TI-84 Plus



Casio fx-9860G