MATHEMATICS (H1) — 8863/01

Paper 1 Suggested Solutions — October/November 2009

1. **Topic: Simultaneous Equations**

$$x + 2y = 3$$
$$x = 3 - 2y \quad \ldots\ldots\ldots\ldots\ldots (1)$$
$$x^2 + xy = 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Sub (1) into (2),

$$(3 - 2y)^2 + (3 - 2y)y = 2$$
$$9 - 12y + 4y^2 + 3y - 2y^2 = 2$$
$$2y^2 - 9y + 7 = 0$$
$$(2y - 7)(y - 1) = 0$$

$2y - 7 = 0$ or $y - 1 = 0$

$y = 3.5$ $\qquad$ $y = 1$

Sub $y = 3.5$ into (1),

$$x = 3 - 2(3.5)$$
$$= -4$$

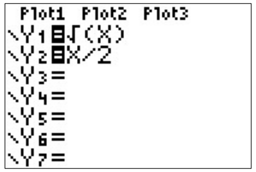
Sub $y = 1$ into (1),

$$x = 3 - 2(1)$$
$$= 1$$

$\therefore x = -4, y = 3.5$ and $x = 1, y = 1$

2. **Topic: Graphs, Integration**

(i)



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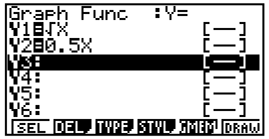

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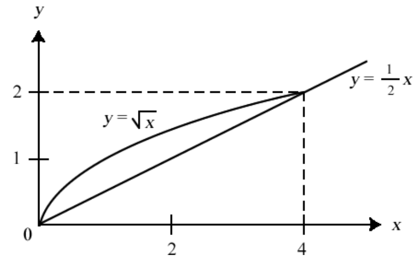
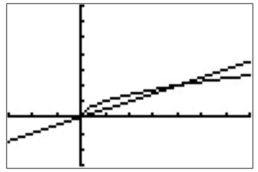




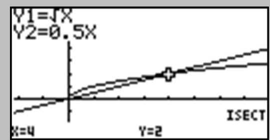
$$y = \sqrt{x} \qquad - (1)$$
$$y = \tfrac{1}{2}x \qquad - (2)$$

Equating (1) & (2):

$$\sqrt{x} = \tfrac{1}{2}x$$
$$x = \tfrac{1}{4}x^2$$
$$x^2 = 4x$$
$$x^2 - 4x = 0$$
$$x(x - 4) = 0$$

$x = 0$ or $x = 4$

$\therefore y = 0$ $\qquad$ $y = 2$



Use GC to check if pts of intersection are correct

$\therefore$ Coordinates of intersection are (0, 0) and (4, 2)

(ii) Area of the region between the two graphs $= \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right)\,\mathrm{d}x$

$= \int_0^4 \left(x^{\frac{1}{2}} - \frac{1}{2}x\right)\,\mathrm{d}x$

$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$

$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{4}\right]_0^4$

$= \frac{4^{\frac{3}{2}}}{\frac{3}{2}} - \frac{4^2}{4}$

$= 5\frac{1}{3} - 4$

$=$ **$1\frac{1}{3}$ unit^2**

3. **Topic: Functions**

(i) $\mathrm{f}(x) = \mathrm{e}^x$

Let $y = \mathrm{e}^x$,

$\ln y = x$

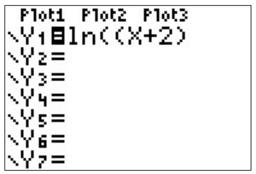
$\therefore \mathrm{f}^{-1}(x) = \ln x, x > 0$

$\Rightarrow \mathrm{h}(x) = \mathrm{f}^{-1}\mathrm{g}(x)$

$= \mathrm{f}^{-1}(x+2)$

$=$ **$\ln(x+2), x > -2$**

(ii)



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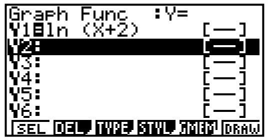

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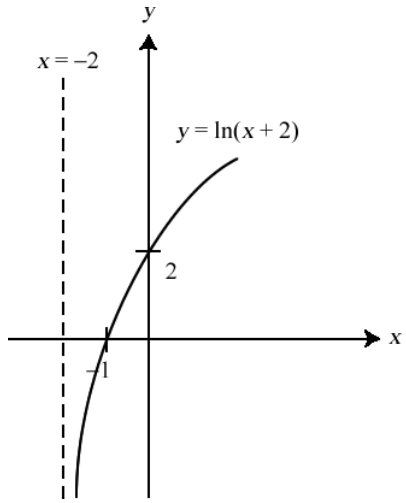
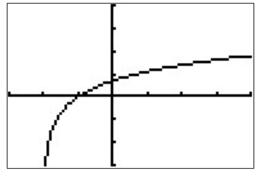




$y = \ln(x+2)$

When $x = 0$, $y = \ln 2$

When $y = 0$, $\ln(x+2) = 0$

$x + 2 = \mathrm{e}^0$

$x = -1$

Equation of asymptote: **$x = -2$**

Coordinates of points that cuts x-axis = **$(-1, 0)$**

Coordinates of points that cuts y-axis = **$(0, \ln 2)$**

(iii) $\mathrm{g}(x) = x + 2$

$\mathrm{g}(-x) = -x + 2$

Given $\mathrm{h}(x) = \mathrm{g}(-x)$:

$\Rightarrow \ln(x+2) = -x + 2$

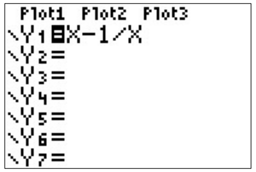
Using G.C., draw $y = \ln(x+2)$ and $y = -x + 2$

$\therefore x = 0.9262$

$\approx$ **0.926 (3 d.p.)**

4. **Topics: Differentiation, Differential Equations**

(i)



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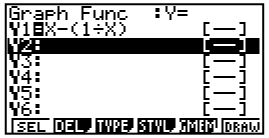

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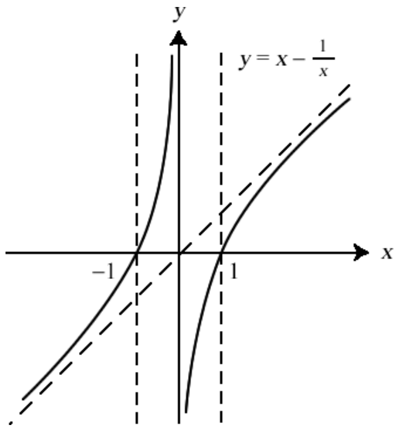
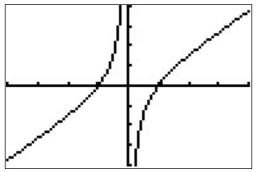




$$y = x - \frac{1}{x}$$

When $y = 0$, $x - \frac{1}{x} = 0$

$$x = \frac{1}{x}$$

$$x^2 = 1$$

$$\boldsymbol{x = 1} \quad \text{or} \quad \boldsymbol{x = -1}$$

(ii)

$$y = x - \frac{1}{x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{1}{x^2}$$

When $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{4}$ and $y = \frac{3}{2}$

$\therefore$ Gradient of the normal $= -1 \div \frac{5}{4}$

$$= \boldsymbol{-\frac{4}{5}}$$

(iii) Equation of the normal at P:

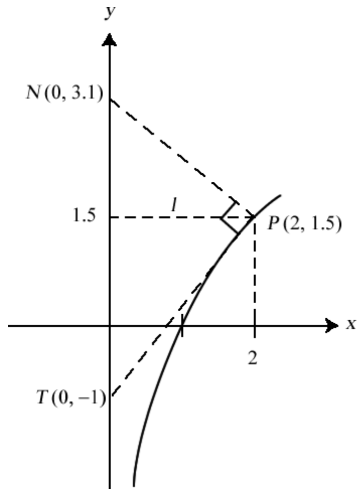
$$y - \frac{3}{2} = -\frac{4}{5}(x - 2)$$

$$y = -\frac{4}{5}x + \frac{31}{10}$$

$$5y = -4x + 15.5$$

$$\boldsymbol{5y + 4x - 15.5 = 0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(iv)



Sub $x = 0$ into (1),

$$5y = 15.5$$
$$y = \frac{31}{10}$$

$\therefore$ Coordinates of $N = \left(0, \frac{31}{10}\right)$

Equation of tangent at P:

$$y - \frac{3}{2} = \frac{5}{4}(x-2)$$
$$y = \frac{5}{4}x - 1 \text{ .... (2)}$$

Sub $x = 0$ into (2), $y = -1$

$\therefore$ Coordinates of $T = (0, -1)$

$$\text{Area of } \Delta PTN = \frac{1}{2} \times l \times TN$$
$$= \frac{1}{2} \times 2 \times \left(\frac{31}{10} + 1\right)$$
$$= \mathbf{4.1\ unit^2}$$

5. **Topic: Differentiation**

$y = 2x^3 - 5x^2 - 4x + 3$ .... (1)

(i) $\frac{dy}{dx} = \mathbf{6x^2 - 10x - 4}$

For stationary points, $\frac{dy}{dx} = 0$

$$6x^2 - 10x - 4 = 0$$
$$3x^2 - 5x - 2 = 0$$
$$(3x+1)(x-2) = 0$$
$$3x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -\frac{1}{3} \qquad x = 2$$

Sub $x = -\frac{1}{3}$ into (1),

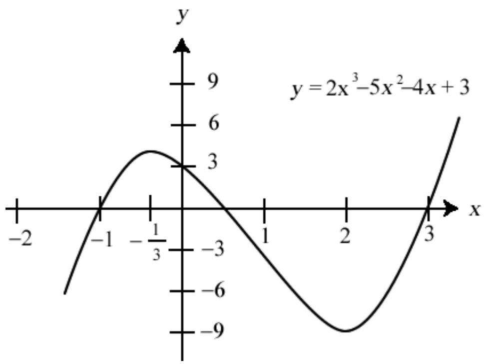
$$y = 2\left(-\frac{1}{3}\right)^3 - 5\left(-\frac{1}{3}\right)^2 - 4\left(-\frac{1}{3}\right) + 3$$
$$= 3\frac{19}{27}$$

Sub $x = 2$ into (1),

$$y = 2(2)^3 - 5(2)^2 - 4(2) + 3$$
$$= -9$$

$\therefore$ **Coordinates of the stationary points on the curve are $\left(-\frac{1}{3}, 3\frac{19}{27}\right)$ and $(2, -9)$**

(ii)



$$\begin{aligned} y &= 2x^3 - 5x^2 - 4x + 3 \\ &= (x+1)(2x^2 - 7x + 3) \\ &= (x+1)(2x-1)(x-3) \end{aligned}$$

When $y = 0$, $\boldsymbol{x = -1, \frac{1}{2}, 3}$

(iii) $2x^3 - 5x^2 - 4x + 3 > 0$

From the graph in (ii), $-1 < x < \frac{1}{2}$, $x > 3$

$2e^{3x} - 5e^{2x} - 4e^x + 3 > 0$

$2(e^x)^3 - 5(e^x)^2 - 4(e^x) + 3 > 0$

$\Rightarrow -1 < e^x < \frac{1}{2}$ $\qquad$ $e^x > 3$

$\Rightarrow 0 < e^x < \frac{1}{2}$ (e^x is always positive) $\qquad$ $\boldsymbol{\therefore x > \ln 3}$

$\boldsymbol{\therefore x < \ln \frac{1}{2}}$

6. **Topic: Probability**

(i) P(the call is for A and A is in the office) $= 0.2 \times 0.7$
$= \mathbf{0.14}$

(ii) P(the researcher being called is in the office)
= P(A is in office or B is in office or C is in office)
= 0.2(0.7) + 0.3(0.6) + 0.5(0.8)
= **0.72**

*Note that this is a conditional probability question.

(iii) Let x : call for C
y : the researcher being called is not in the office

$$\begin{aligned} \therefore P(x \mid y) &= \frac{P(x \cap y)}{P(y)} \\ &= \frac{0.5(0.2)}{1 - P(y')} \\ &= \frac{0.5(0.2)}{1 - 0.72} \quad \text{From (ii)} \\ &\approx 0.3571 \\ &\approx \mathbf{0.357 \ (3 \ sig. \ fig.)} \end{aligned}$$

7. **Topic: Probability**

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\frac{17}{30} = \frac{1}{3} + \frac{2}{5} - P(A \cap B)$

$\mathbf{P(A \cap B) = \frac{1}{6}}$

(ii) If A and B are not independent ⇒ $P(A)\cdot P(B) = P(A \cap B)$

L.H.S.: $P(A)\cdot P(B) = \frac{1}{3} \times \frac{2}{5}$

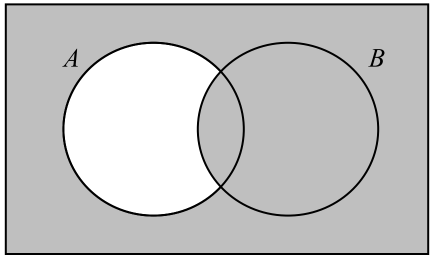
$= \frac{2}{15}$

R.H.S.: $P(A \cap B) = \frac{1}{6}$

$P(A)\cdot P(B) \neq P(A \cap B)$

∴ A and B are not independent. **(Shown)**

(iii)



$P(A' \cup B) = 1 - P(A) + P(A \cap B)$

From (i)

$= 1 - \frac{1}{3} + \frac{1}{6}$

$= \mathbf{\frac{5}{6}}$

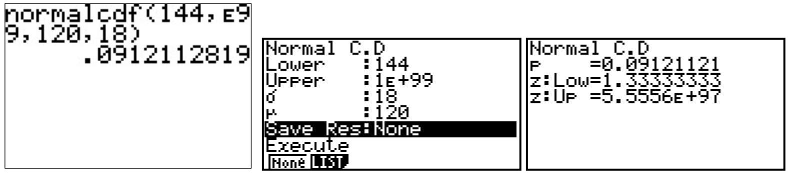
8. **Topic: Normal Distribution, Hypothesis Testing**

Let the random variable X be the lifetime of a component.

$X \sim N(120, 18^2)$

(i) $P(x > 144) = 0.091211$

$\approx$ **0.0912 (3 sig. fig.)**



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(ii) $P(X_1 > 144 \text{ and } X_2 < 144) + P(X_1 < 144 \text{ and } X_2 > 144)$

$= 0.091211(1 - 0.091211) + (1 - 0.091211)(0.091211)$

$= 0.16578$

$\approx$ **0.166 (3 sig. fig.)**

P(1st comp lifetime > 144 AND 2nd component lifetime < 144) OR P(1st component lifetime < 144 AND 2nd component lifetime > 144)

$H_o : \mu = 120$ days

$H_1 : \mu > 120$ days

$\bar{X} \sim N\left(120, \frac{18^2}{50}\right)$

Since population mean is tested, divide variance by population size

Since σ is known and n is large (≥ 50), a z-test is used.

Using G.C., $z_{test} = 1.5713$

$z_{test} = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$

$= \frac{124-120}{\frac{18}{\sqrt{50}}}$

$= 1.571$ ($< z_{critical}$)

Since $z_{test} < z_{critical}$, do not reject H_o.

There is insufficient evidence, at the 5% level of significance, to support the company's claim that the mean lifetime is longer than for the old components.

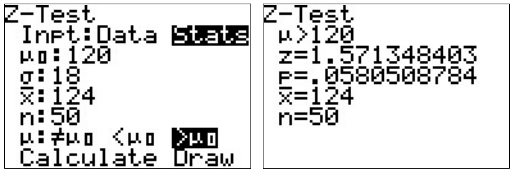
ALTERNATE APPROACH

$H_o : \mu = 120$ days
$H_1 : \mu > 120$ days

$\bar{X} \sim N\left(120, \frac{18^2}{50}\right)$
Since σ is known and n is large (≥ 50), a z- test is used.

Using G.C., p-value = 0.05805

Since p-value > 0.05, do not reject H_o.
There is insufficient evidence, at the 5% level of significance, to support the company's claim that the mean lifetime is longer than for the old components.



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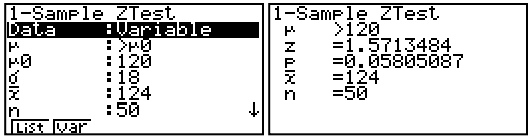

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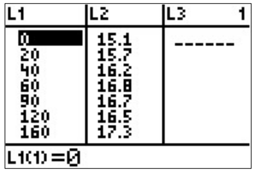
9. **Topic: Correlation Coefficient and Linear Regression**

(i)



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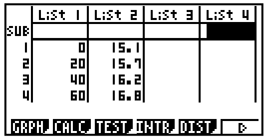

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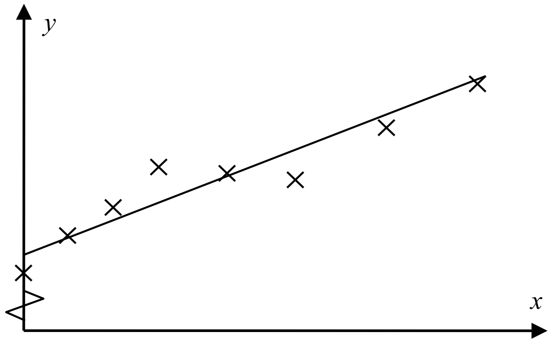
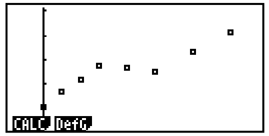
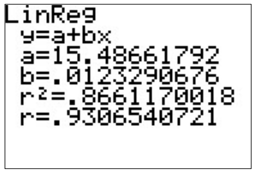




(ii)



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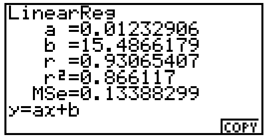

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Product moment coefficient, r = **0.931**

The *r* value of 0.931 indicates a reasonably strong positive linear correlation between the volume of the liquid nutrient added and the total weight of fruits per tree.

(iii) From G.C.: $y = 15.486 + 0.01232x$
$\approx$ **$15.5 + 0.0123x$**

(iv) When $x = 135$, $y = 15.5 + 0.0123(135)$
$= 17.1605$
$\approx$ **17.2 kg**

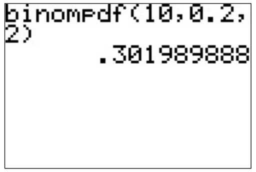
(v) **Since the volume of liquid nutrient needed for 20 kg of fruit is estimated through extrapolating the data beyond 18.1 kg, it might be unsuitable to use the equation.**

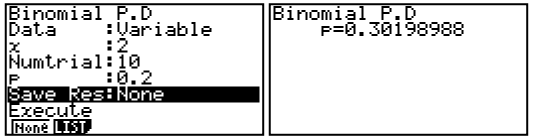
10. **Topic: Binomial and Normal Distributions**

(i) Let the random variable X be the number of candidates who fail the piano exam.

$X \sim B(10, 0.2)$

$\therefore P(X = 2) =$ **0.302**





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(ii) Probability of candidates who pass the piano examination and awarded a distinction

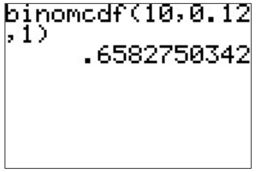
$= 0.15(1 - 0.2)$

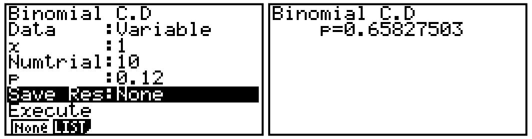
$= 0.12$

Let the random variable Y be the number of candidates who pass the piano examination and awarded a distinction.

$Y \sim B(10, 0.12)$

$P(Y < 2) = P(Y \leq 1)$
$= 0.65827$
$\approx$ **0.658 (3 sig. fig.)**





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(iii) Let the random variable W be the number of candidates who fail the piano examination.

$W \sim B(50, 0.2)$

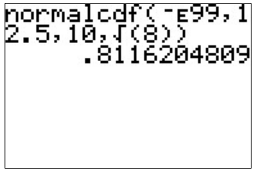
Since $np = 10$ (> 5) and $nq = 40$ (> 5),

We can approximate $W \sim N(10, 8)$ using normal distribution

$\therefore P(W \leq 12) = P(W < 12.5)$ [Continuity Correction]

$=$ **0.812**

Use continuity correction to approximate a discrete distribution (i.e binomial) by a continuous distribution (i.e normal).



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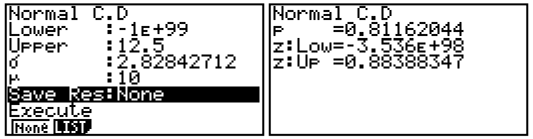

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11. **Topic: Sampling and Hypothesis Testing**

(a) (i) A systematic random sample of 8 may be obtained by first arranging the claims in order of the time in which they are received and then selecting every 9th claim from the stack, from a random starting point in the stack.

(ii) Choosing the first of claims received would not give a good indication as the claims that arrive the earliest will tend to be of relatively low-value, since less time is needed to access the claim amounts when less items have been damaged by the flood. Hence a systematic random sample will give a better indication of the value of the 72 claims.

(b) (i) Unbiased estimate of population mean:

$$\bar{x} = \frac{\sum x}{n}$$
$$= \frac{\sum(x-1000)}{n} + 1000$$
$$= \frac{5320}{120} + 1000$$
$$= \mathbf{1044\frac{1}{3}}$$

Unbiased estimate of population variance:

$$S^2 = \frac{n}{n-1}\left\{\frac{\sum(x-1000)^2}{n} - \left[\frac{\sum(x-1000)}{n}\right]^2\right\}$$
$$= \frac{120}{119}\left\{\frac{8282000}{120} - \left(\frac{5320}{120}\right)^2\right\}$$
$$\approx 67614.67$$
$$\approx \mathbf{67600\ (3\ sig.\ fig.)}$$

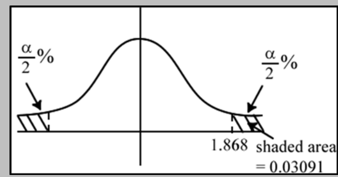
(ii) An unbiased estimate is an estimate for a parameter of a distribution whose expected value is equal to the true value of the parameter being estimated.

(iii) H_o: $\mu = 1000$

H_1: $\mu \neq 1000$

$$\bar{X} \sim N\left(1000, \frac{67614.67}{120}\right)$$
$$z = \frac{\bar{x}-\mu}{\frac{S}{\sqrt{n}}}$$
$$= \frac{1044\frac{1}{3}-1000}{\sqrt{\frac{67614.67}{120}}}$$
$$\approx 1.86753$$

For two-tailed test:

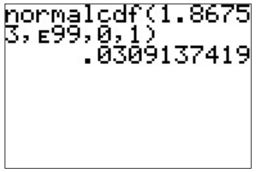


From G.C., $\alpha\% = 2 \times 0.03091$

$= 0.06182 \times 100\%$

$\approx 6.18\%$

$\therefore \alpha = 6.18$



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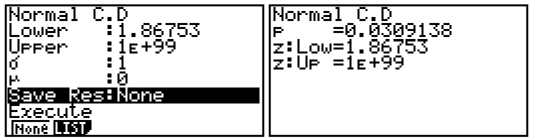

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Set of values of α: $|\boldsymbol{\alpha}| \geq \mathbf{6.18}$

12. **Topic: Normal Distribution**

(a) Let $X \sim N(\mu, \sigma^2)$

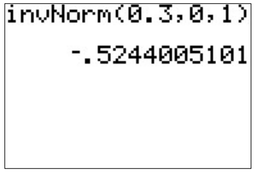
$$Z = \frac{x-\mu}{\sigma} \sim N(0, 1)$$

Using G.C.,

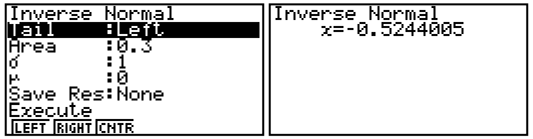
$$\Rightarrow \frac{22-\mu}{\sigma} = -0.52440 \text{ ………………… (1)}$$

$$\Rightarrow \frac{29-\mu}{\sigma} = 0.84162 \text{ …………………… (2)}$$

$\frac{22-\mu}{\sigma}$:



TI-84 Plus



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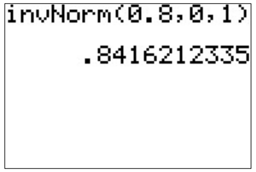
$\frac{29-\mu}{\sigma}$:



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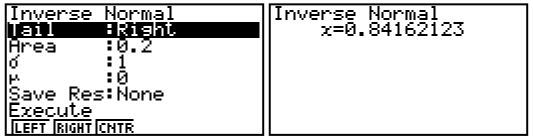

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(2) – (1),

$$\frac{7}{\sigma} = 0.84162 + 0.52440$$

$$\sigma = 5.1243$$

$$\approx 5.12 \text{ (3 sig. fig.)}$$

Sub $\sigma = 5.1243$ into (1),

$$22 - \mu = -0.52440(5.1243)$$

$$\mu \approx 24.687$$

$$\approx 24.7 \text{ (3 sig. fig.)}$$

$\therefore$ **$\sigma = 5.12$** and **$\mu = 24.7$**

(b) (i) Let the random variable X be the mass of Apple and the random variable Y be the mass of Nectarines.

$$X \sim N(0.15, 0.03^2)$$

$$Y \sim N(0.07, 0.02^2)$$

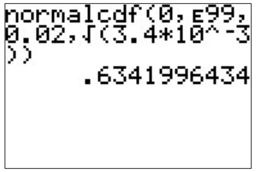
$$X_1 + X_2 - (Y_1 + \ldots + Y_4) \sim N[\,0.15(2) - 0.07(4), (0.03)^2(2) + (0.02)^2(4)]$$

$$\sim N(0.02, 3.4 \times 10^{-3})$$

$$P(X_1 + X_2 > Y_1 + Y_2 + Y_3 + Y_4) = P[X_1 + X_2 - (Y_1 + \ldots + Y_4) > 0]$$

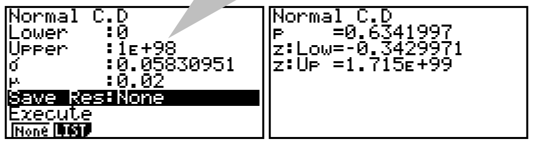
$$= 0.63419$$

$$\approx \mathbf{0.634 \text{ (3 sig. fig.)}}$$



TI-84 Plus

Use 1E+98 instead of 1E+99 as the latter is too large



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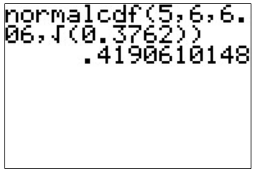
(ii) Let $C = 9(X_1 + X_2) + 12(Y_1 + Y_2 + Y_3 + Y_4)$

$C \sim N[9(0.15 \times 2) + 12(0.07 \times 4),\ 9^2(2)(0.03)^2 + 12^2(4)(0.02)]$

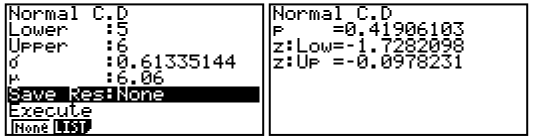
$\sim N(6.06,\ 0.3762)$

$P(5 < C < 6) = 0.41906$

$\approx$ **0.419 (3 sig. fig.)**



TI-84 Plus



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