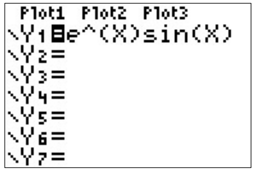
MATHEMATICS (H2) 9740/02

Paper 2 Suggested Solutions October/November 2008

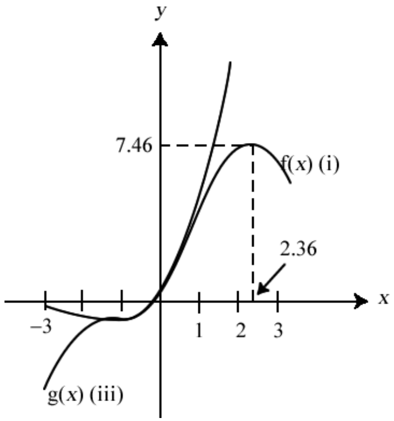
1. **Topic : Functions and Graphs**

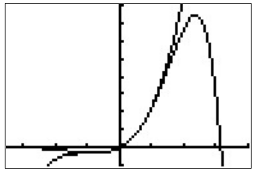
(i)



TI-84 Plus

Casio fx-9860G





(ii) $\mathrm{f}(x) = e^x \sin x$

$= (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots)(x - \frac{x^3}{6} + \ldots)$

$= x - \frac{x^3}{6} + x^2 + \frac{x^3}{2} + \ldots$

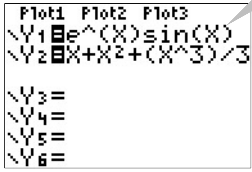
$= \boldsymbol{x + x^2 + \frac{x^3}{3} + \ldots}$

$\mathrm{g}(x) = x + x^2 + \frac{x^3}{3}$

Refer to formula list under Maclaurin’s expansion

(iii)

Ensure calculator’s mode is in Radians.



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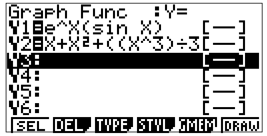

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Refer to part (i)

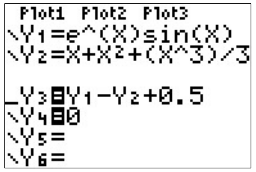
(iv) $|\mathrm{f}(x) - \mathrm{g}(x)| < 0.5$

$\left|e^x \sin x - (x + x^2 + \frac{x^3}{3})\right| < 0.5$

$-0.5 < e^x \sin x - (x + x^2 + \frac{x^3}{3}) < 0.5$

$e^x \sin x - (x + x^2 + \frac{x^3}{3}) + 0.5 > 0$ or $e^x \sin x - (x + x^2 + \frac{x^3}{3}) - 0.5 < 0$

$x < 1.56$ $\quad$ $x > -1.96$



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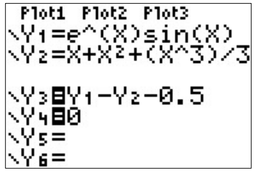

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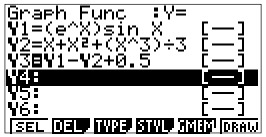

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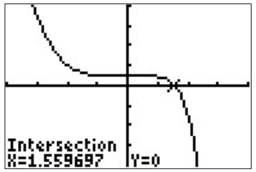
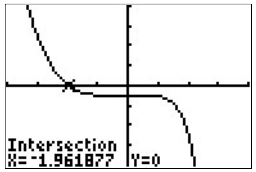




$\therefore\ \boldsymbol{-1.96 \le x \le 1.56}$

2. **Topic : Definite Integrals**

(i) $y^2 = x(1-x)^{\frac{1}{2}}$

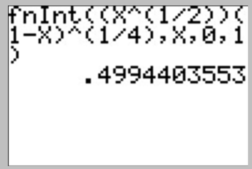
$y = x^{\frac{1}{2}}(1-x)^{\frac{1}{4}}$

Area of $R = \mathbf{2\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{4}}\,dx}$

$= 2\,(0.49944)$

$= 0.9988$

$\approx$ **0.999 (3 sig. fig.)**



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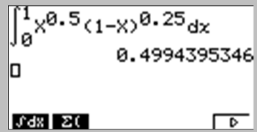

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(ii) Let $u = 1-x$

$\therefore du = -dx$

When $x = 0$, $u = 1$

When $x = 1$, $u = 0$

Volume $= \pi\int_0^1 y^2\,dx$

$= \pi\int_0^1 x\sqrt{1-x}\,dx$

$= \pi\int_1^0 -(1-u)\sqrt{u}\,du$

$= \pi\int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}})\,du$

$= \pi\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1$

$= \pi\left[\frac{2}{3} - \frac{2}{5}\right]$

$= \mathbf{\frac{4}{15}\pi}$ **units^3**

Reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $y^2 = x\sqrt{1-x}$

$2y\frac{dy}{dx} = x\left(\frac{1}{2}\right)(1-x)^{-\frac{1}{2}}(-1) + \sqrt{1-x}$

$2y\frac{dy}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}}[-x + 2(1-x)]$

$\frac{dy}{dx} = \left(\frac{1}{4}\right)\frac{2-3x}{y\sqrt{1-x}}$

To find max point, let $\frac{dy}{dx} = 0$

$\left(\frac{1}{4}\right)\frac{2-3x}{y\sqrt{1-x}} = 0$

$2 - 3x = 0$

$\therefore \mathbf{x = \frac{2}{3}}$

3. **Topic : Complex numbers (polar form)**

(a) $w = re^{i\theta}$

$w^* = re^{i(-\theta)}$

$p = \frac{w}{w^*}$

$= \frac{r\,e^{i\theta}}{r\,e^{i(-\theta)}}$

$p = e^{i2\theta}$

$\mathbf{|p| = 1, \arg(p) = 2\theta}$

$p^5 = (e^{i2\theta})^5$

$= e^{i10\theta}$

$= \cos 10\theta + i\sin 10\theta$

Given that p^5 is positive and real,

$p^5 = \cos 10\theta + i\,(0)$

$\Rightarrow \sin 10\theta = 0$

Basic $\angle = 0$

$10\theta = 2\pi, 4\pi$

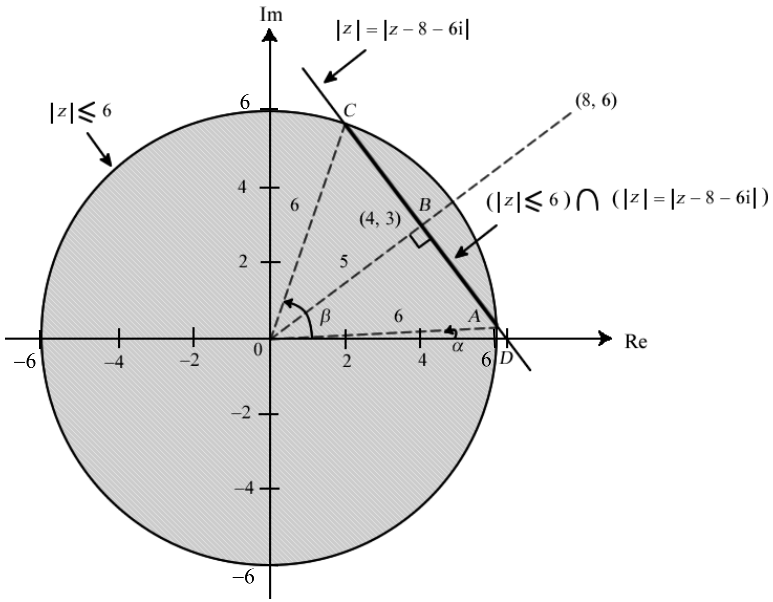
$\therefore \theta = \frac{2\pi}{10}, \frac{4\pi}{10}$

$= \mathbf{\frac{\pi}{5}, \frac{2\pi}{5}}$

For a complex number $z = x + iy$, when z is real, imaginary part $y = 0$

Given that $0 < \theta < \frac{\pi}{2}$, $0 < 10\theta < 5\pi$

(b) (i)



(ii) $\cos\angle AOB = \frac{5}{6}$

$\angle AOB = \cos^{-1}\left(\frac{5}{6}\right)$

$= 0.58569$

$\sin\angle BOD = \frac{3}{5}$

$\angle BOD = \sin^{-1}\left(\frac{3}{5}\right)$

$= 0.64350$

$\angle COB = \angle AOB$

$= 0.58569$

$\therefore$ least possible value of arg z $= \alpha$

$= \angle BOD - \angle AOB$

$= 0.64350 - 0.58569$

Smallest arg z occurs at $\angle AOD = \alpha$

$= 0.05781$

$\approx$ **0.058 rad (3 d.p.)**

$\therefore$ greatest possible value of arg z $= \beta$

$= \angle COB + \angle AOB + \alpha$

$= 0.58569 + 0.58569 + 0.05781$

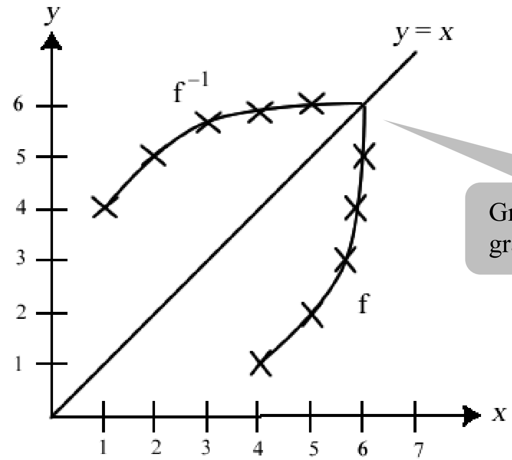
$= 1.2292$

$\approx$ **1.229 rad (3 d.p.)**

Greatest arg z occurs at $\angle COD = \beta$

4. **Topic : Functions and inverse functions**

(i)



Graph of f^{-1} is a reflection of graph of f in the line $y = x$.

(ii) Let

$$\begin{aligned} y &= (x-4)^2+1 \\ (x-4)^2 &= y-1 \\ x-4 &= \pm\sqrt{y-1} \\ x &= \pm\sqrt{y-1}+4 \\ \therefore \mathbf{f^{-1}(x)} &= \mathbf{4+\sqrt{x-1}\,,\ x>1} \\ \mathbf{D_{f^{-1}}} &= \mathbf{(1,\infty)} \end{aligned}$$

$D_{f^{-1}} = R_f$

(iv) Equation of the line $\Rightarrow$ $\boldsymbol{y = x}$

$$\begin{aligned} f(x) &= f^{-1}(x) = x \\ (x-4)^2+1 &= x \\ x^2-8x+16+1-x &= 0 \\ x^2-9x+17 &= 0 \end{aligned}$$

Common point of intersection of the 3 graphs

$$\begin{aligned} x &= \frac{9 \pm \sqrt{(-9)^2-4(1)(17)}}{2} \\ &= \frac{9 \pm \sqrt{81-68}}{2} \\ &= \frac{9 \pm \sqrt{13}}{2} \\ &= \frac{9+\sqrt{13}}{2} \text{ or } \frac{9-\sqrt{13}}{2} \text{ (rej.)} \end{aligned}$$

$$\therefore \boldsymbol{x = \frac{9+\sqrt{13}}{2}}$$

5. **Topic : Sampling**

Call an assembly of the 950 pupils.

Start from a randomly selected student and pick every $\frac{950}{50} = 19^{th}$ student.

As the different pupils belong to different groups and organizations that will use different sports facilities for different activities, a stratified sample put together from random samples taken from each group will provide a more accurate representation of the pupil population without any possible bias towards any groups.

6. **Topic : Hypothesis Testing**

$$\begin{aligned} \bar{x} &= \frac{\sum x}{n} \\ &= \frac{1026}{15} \\ &= 68.4 \end{aligned}$$

$$\begin{aligned} S^2 &= \frac{1}{n-1}\left[\sum x^2 - \frac{(\sum x)^2}{n}\right] \\ &= \frac{1}{14}\left[77265.90 - \frac{1026^2}{15}\right] \\ &= 506.25 = (22.5)^2 \end{aligned}$$

$$\bar{X} \sim N(78, \frac{S^2}{n})$$

$$\bar{X} \sim N(78\,, \frac{506.25}{15})$$

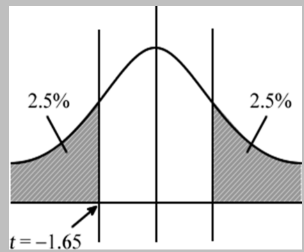
$H_0 :\ \mu = 78$

$H_1 :\ \mu \neq 78$

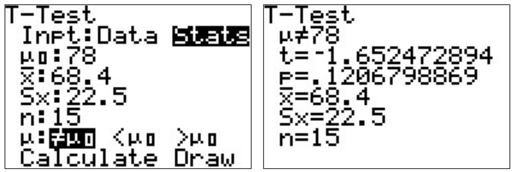
From G.C,

$t = -1.6524$

$p = 0.121$



Since $p = 0.121 \geq 0.05$, we do not reject H_0. We conclude that at the 5% level, there is insufficient of evidence to suggest that the mean mass of calcium in a bottle has changed.



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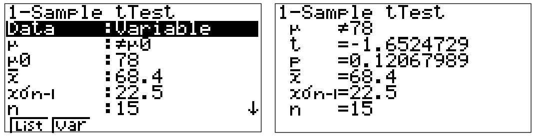

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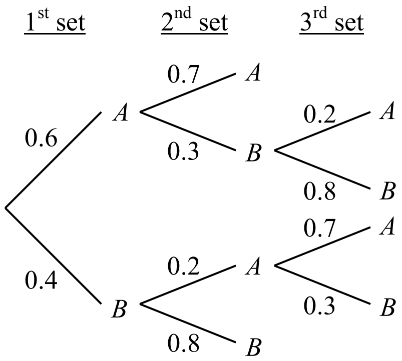
7. **Topic : Probability**



(i) P(A wins in the second set) $= 0.6\,(0.7) + 0.4\,(0.2)$
$= \mathbf{0.5}$ [*AA* + *BA*]

(ii) P (A wins the match) $= 0.6\,(0.7) + 0.4\,(0.2)(0.7) + 0.6\,(0.3)(0.2)$
$= 0.42 + 0.056 + 0.036$
$= \mathbf{0.512}$ [*AA* + *BAA* + *ABA*]

*Note that this is a conditional probability question.

(iii) P(B won the first set | A won the match)

$$= \frac{\text{P}(B \text{ wins 1st set and } A \text{ wins the match})}{\text{P}(A \text{ wins the match})}$$

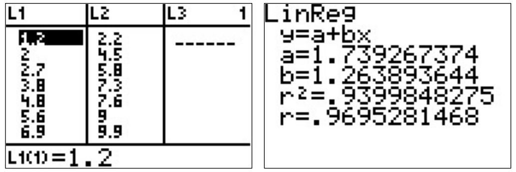
$$= \frac{0.4(0.2)(0.7)}{0.512}$$ [*BAA*; from (ii)]

$$= \mathbf{\frac{7}{64}}$$

8. **Topic : Correlation coefficient and linear regression**

(i) From G.C, r = **0.970 (3 sig. fig.)**

Since r is close to 1, this indicates that there is a strong positive linear correlation between x and t.



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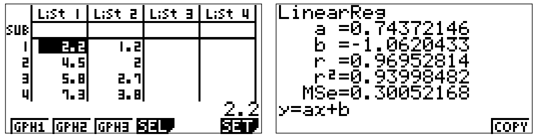

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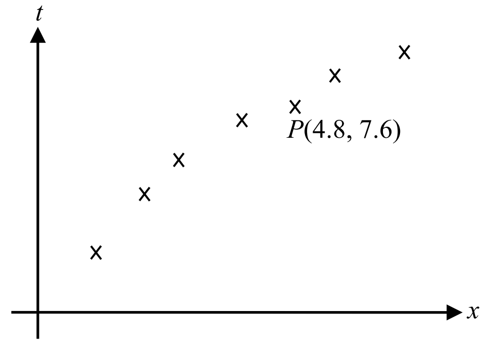
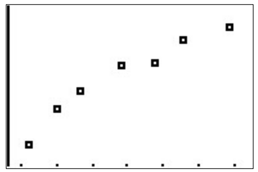
(ii)



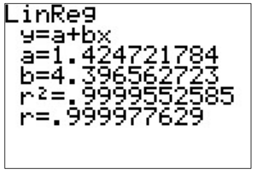


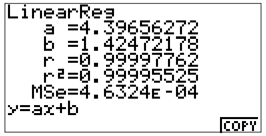
(iii) **As the remaining points' scattering pattern appears to follow that of a logarithmic data set, the scatter diagram may be modeled by a straight line $t = a + b \ln x$.**

(iv) Using G.C,

$t = 1.42 + 4.40 \ln x$

$\therefore$ **a = 1.42, b = 4.40**





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Remove coordinates of P and ln values of x.

(v) $t = 1.42 + 4.40 \ln x$

When $x = 4.8$,

$t = 1.42 + 4.40 \ln 4.8$

$= $ **8.32 (3 sig. fig.)**

(vi) **Since $x = 8$ is out of the given data set's range, the value of t obtained may not be accurate.**

9. **Topic : Poisson Distribution**

Let r.v. X = number of grand pianos sold in a week

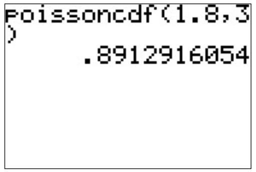
$X \sim \text{Po}(1.8)$

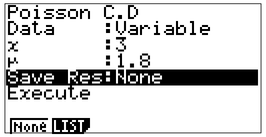
$P(X \geq 4) = 1 - P(X \leq 3)$

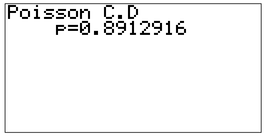
$= 1 - 0.89129$

$= 0.1087$

$\approx$ **0.109 (3 sig. fig.)**







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Let r.v. Y = number of upright pianos sold in a week , $Y \sim \text{Po}(2.6)$

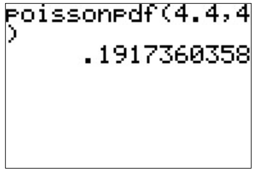
$X + Y$ = total number of pianos sold in a week

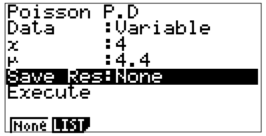
$X + Y \sim \text{Po}(1.8 + 2.6)$

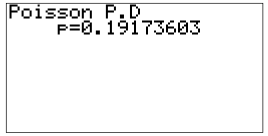
$\sim \text{Po}(4.4)$

$P(X + Y = 4) = 0.1917$

$\approx$ **0.192 (3 sig. fig.)**







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Mean number of pianos sold in a year $= 50 \times 1.8$

$= 90$

Let A = number of grand pianos sold in a year.

$A \sim \text{Po}(90)$

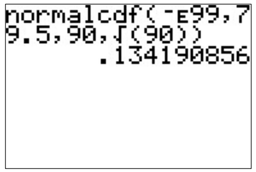
Since mean > 10, $A \sim N(90, 90)$ approximately

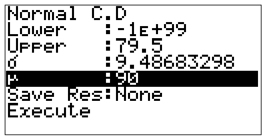
Use continuity correction to approximate a discrete distribution (i.e Poisson) by a continuous distribution (i.e normal).

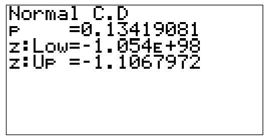
$P(A < 80) = P(A < 79.5)$ [Continuity Correction]

$= 0.1341$

$\approx$ **0.134 (3 sig. fig.)**







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The sales of the pianos are likely to follow demand trends across the different months of the year. Hence a piano sale should not be considered random event within the interval of a year.

10. **Topic : Combinations**

(i) $^3C_2\ ^4C_3\ ^5C_3 = \mathbf{120}$

> Choose 2 from 3 K
> 3 from 4 L
> 3 from 5 M

(ii) $^9C_8 = \mathbf{9}$

> Choose 8 from the total of 9 from L & M

(iii) $^5C_4\ ^7C_4 + {^5C_5}\ ^7C_3 = 175 + 35$
$= \mathbf{210}$

> 4 from M and 4 from remaining 7 $(K + L)$
> **OR**
> 5 from M and 3 from remaining 7 $(K + L)$

(iv) $^{12}C_8 - {^9C_8} - {^8C_8} - 0 = 495 - 9 - 1 - 0$
$= \mathbf{485}$

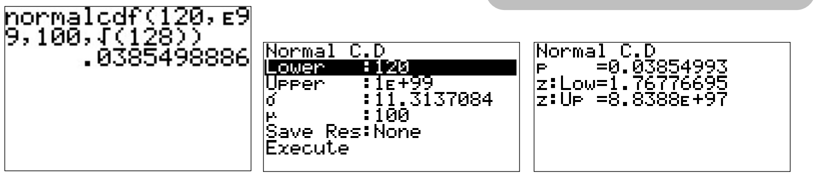
> $^{12}C_8$: total no. of ways with no restriction
> $^9C_8$: group without K
> $^8C_8$: group without L
> 0: group without M (impossible as $K + L = 7$)

11. **Topic : Normal distributions**

(i) $X_1 + X_2 \sim N(50 \times 2, 8^2 \times 2)$
$\sim N(100, 128)$

> $X_1 = X_2 \sim N(\mu, \sigma^2)$
> $E(X_1 + X_2) = \mu + \mu = 2\mu$
> $Var(X_1 + X_2) = \sigma^2 + \sigma^2 = 2\sigma^2$

$P(X_1 + X_2 > 120) = 0.03854$
$\approx$ **0.0385 (3 sig. fig.)**



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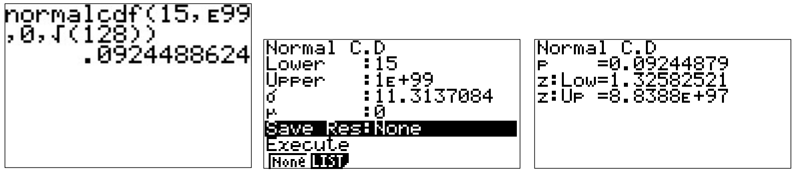
(ii) $X_1 - X_2 \sim N(0, 128)$

> $E(X_1 - X_2) = \mu - \mu = 0$
> $Var(X_1 - X_2) = \sigma^2 + \sigma^2 = 2\sigma^2$

$P(X_1 > X_2 + 15) = P(X_1 - X_2 > 15)$
$=$ **0.0924 (3 sig. fig.)**



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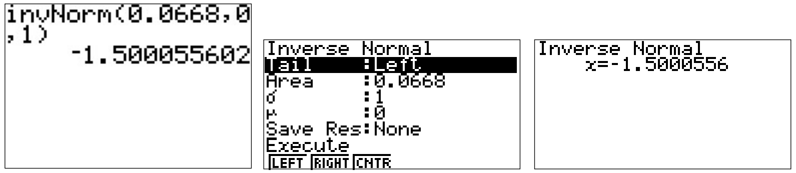
Consider, $Y \sim N(\mu, \sigma^2)$

> Change to standard normal
> $X \sim N(\mu, \sigma^2)$
> $z = \frac{X-\mu}{\sqrt{\sigma^2}}$
> $= \frac{X-\mu}{\sigma} \sim N(0, 1)$

$P(Y < 74) = 0.0668$

$P(Z < \frac{74-\mu}{\sigma}) = 0.0668$

$\Rightarrow \frac{74-\mu}{\sigma} = -1.5 \quad -(1)$



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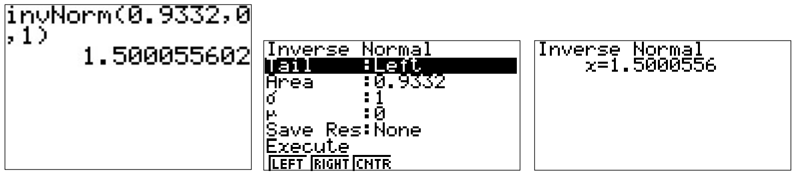
$P(Y > 146) = 0.0668$
$P(Y < 146) = 1 - 0.0668$
$= 0.9332$

$P(Z < \frac{146-\mu}{\sigma}) = 0.9332$

$\Rightarrow \frac{146-\mu}{\sigma} = 1.5 \quad -(2)$



TI-84 Plus **Casio fx-9860G**

(1) – (2),

$$-\frac{72}{\sigma} = -3$$
$$\sigma = 24$$

sub $\sigma = 24$ into (1),

$$74 - \mu = -1.5(24)$$
$$\mu = 110$$

$\therefore \mu = 110, \sigma = 24$

$$\therefore \text{E}(Y) = 110$$
$$\text{Var}(Y) = 24^2 = 576$$

$$\text{E}(aX + b) = 110$$
$$a\text{E}(X) + b = 110$$
$$50a + b = 110$$
$$b = 110 - 50a \quad -(3)$$

From qn, E(X) = 50

$$\text{Var}(aX + b) = 24^2$$
$$a^2 \text{Var}(X) = 576$$
$$a^2(64) = 576$$
$$a^2 = 9$$
$$a = 3 \text{ or } -3 \text{ (reject)}$$

sub $a = 3$ into (3),

$$b = 110 - 50(3) = -40$$

$\therefore$ **$a = 3$, $b = -40$**