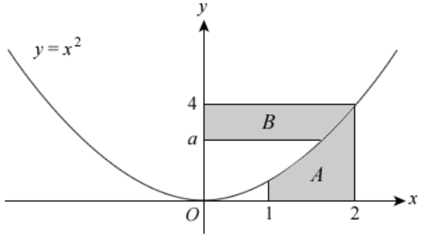
MATHEMATICS (H2) 9740/01

Paper 1 Suggested Solutions

October/November 2008

1. **Topic: Definite Integrals**

From the diagram:



Area $A = \int_1^2 y\,\mathrm{d}x = \int_1^2 x^2\,\mathrm{d}x$…… (1)

Area $B = \int_a^4 x\,\mathrm{d}y = \int_a^4 \sqrt{y}\,\mathrm{d}y$

$= \int_a^4 y^{\frac{1}{2}}\,\mathrm{d}y$…… (2)

Equating (1) & (2):

$$\int_1^2 x^2\,\mathrm{d}x = \int_a^4 y^{\frac{1}{2}}\,\mathrm{d}y$$

$$\left[\frac{x^3}{3}\right]_1^2 = \left[\frac{2}{3}y^{\frac{3}{2}}\right]_a^4$$

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$$

$$\frac{(2)^3}{3} - \frac{(1)^3}{3} = \frac{2}{3}(4)^{\frac{3}{2}} - \frac{2}{3}(a)^{\frac{3}{2}}$$

$$\frac{8}{3} - \frac{1}{3} = \frac{16}{3} - \frac{2}{3}a^{\frac{3}{2}}$$

$$a^{\frac{3}{2}} = \frac{9}{2}$$

$$\therefore a = \left(\frac{9}{2}\right)^{\frac{2}{3}}$$

$\approx$ **2.73 (3 sig. fig.)**

2. **Topic: Summation of Series (Mathematical Induction)**

Let P_n denotes the statement "$S_n = \frac{1}{6}n(n+1)(4n+5), \forall\, n \in \mathbb{Z}^+$".

When $n = 1$,

L.H.S. $= S_1 = u_1 = 3$

R.H.S. $= \frac{1}{6}(1)(1+1)(4+5) = 3$

$\Rightarrow$ L.H.S. = R.H.S.

$\therefore P_1$ is true.

Assume P_k is true i.e. $S_k = \frac{1}{6}k(k+1)(4k+5)$, for some $k \in \mathbb{Z}^+$

To show that P_{k+1} is also true i.e. $S_{k+1} = \frac{1}{6}(k+1)(k+2)(4k+9)$,

$$\begin{aligned}
\text{L.H.S.} &= S_{k+1} \\
&= S_k + u_{k+1} \\
&= \overbrace{\tfrac{1}{6}k(k+1)(4k+5)}^{S_k = \frac{1}{6}k(k+1)(4k+5)} + \underbrace{(k+1)[2(k+1)+1]}_{u_{k+1} = (k+1)[2(k+1)+1]} \\
&= \tfrac{1}{6}(k+1)[k(4k+5) + 6(2k+3)] \\
&= \tfrac{1}{6}(k+1)(4k^2 + 5k + 12k + 18) \\
&= \tfrac{1}{6}(k+1)(4k^2 + 17k + 18) \\
&= \tfrac{1}{6}(k+1)(k+2)(4k+9) \\
&= \text{R. H. S.}
\end{aligned}$$

Bring out the $\frac{1}{6}(k+1)$ factor since it's found on the R.H.S.

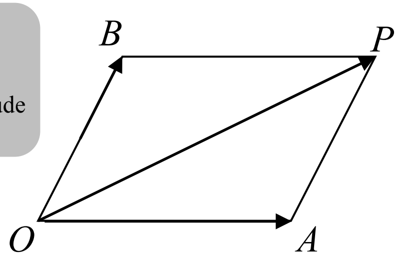
$\therefore P_{k+1}$ is also true if P_k is true.

Since P_1 is true and P_{k+1} is true if P_k is true, by mathematical induction, P_n is true $\forall\, n \in \mathbb{Z}^+$

3. **Topic: Vectors**

(i) $$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA}+\overrightarrow{AP}\\ &= \overrightarrow{OA}+\overrightarrow{OB}\\ &= \begin{pmatrix}1\\4\\-3\end{pmatrix}+\begin{pmatrix}5\\-1\\0\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{6}\\\mathbf{3}\\\mathbf{-3}\end{pmatrix}\end{aligned}$$

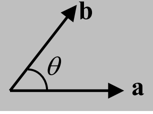
Equal Vectors:
$\overrightarrow{OB}=\overrightarrow{AP}$
∵ same magnitude & direction



(ii) $$\begin{aligned}\cos\angle AOB &= \frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}\\ &= \frac{\begin{pmatrix}1\\4\\-3\end{pmatrix}\cdot\begin{pmatrix}5\\-1\\0\end{pmatrix}}{\sqrt{1^2+4^2+(-3)^2}\sqrt{5^2+(-1)^2+0^2}}\\ &= \frac{(1)(5)+(4)(-1)+(-3)(0)}{\sqrt{26}\sqrt{26}}\\ &= \frac{1}{26}\\ \angle AOB &= \cos^{-1}\left(\frac{1}{26}\right)\\ &= 87.795^\circ\\ &\approx \mathbf{87.8^\circ\ (3\ sig.\ fig.)}\end{aligned}$$

Scalar Product of two vectors **a** and **b**:

$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$



(iii) Area of parallelogram *OAPB*

$$\begin{aligned}&= \left|\overrightarrow{OA}\times\overrightarrow{OB}\right|\\ &= \left|\begin{pmatrix}1\\4\\-3\end{pmatrix}\times\begin{pmatrix}5\\-1\\0\end{pmatrix}\right|\\ &= \left|\begin{pmatrix}-3\\-15\\-21\end{pmatrix}\right|\\ &= \sqrt{(-3)^2+(-15)^2+(-21)^2}\\ &= \sqrt{675}\\ &= \mathbf{15\sqrt{3}\ units^2}\end{aligned}$$

$$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&4&-3\\5&-1&0\end{vmatrix}=\begin{vmatrix}4&-3\\-1&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&-3\\5&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&4\\5&-1\end{vmatrix}\mathbf{k}$$
$$=-3\mathbf{i}-15\mathbf{j}-21\mathbf{k}$$

N.B. Final answer expressed in surd form as question asks for *exact* area.

4. **Topics: Differentiation, Differential Equations**

(i) $$\begin{aligned}\frac{dy}{dx} &= \frac{3x}{x^2+1}\\ \int dy &= \int\frac{3x}{x^2+1}\,dx\\ y &= \frac{3}{2}\int\frac{2x}{x^2+1}\,dx\\ &= \boldsymbol{\frac{3}{2}\ln|x^2+1|+c}\end{aligned}$$

Express in $\frac{f'(x)}{f(x)}$

$$\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+c$$

(ii) Sub $y=2$, $x=0$ into $y=\frac{3}{2}\ln|x^2+1|+c$,

$2=\frac{3}{2}\ln|0^2+1|+c\Rightarrow c=2$

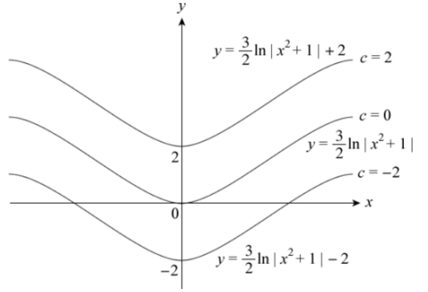
$\therefore\ \boldsymbol{y=\frac{3}{2}\ln|x^2+1|+2}$

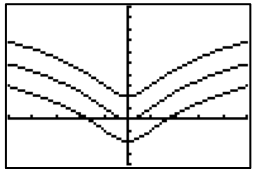
(iii) As $x\to+\infty$, $\frac{dy}{dx}\to 0^+$

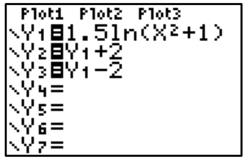
As $x\to-\infty$, $\frac{dy}{dx}\to 0^-$

∴ **the gradient of every solution curve tends to be horizontal as $x\to\pm\infty$.**

(iv)







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 Plot1 Plot2 Plot3
\Y1■1.5ln(X²+1)
\Y2■Y1+2
\Y3■Y1-2
\Y4=
\Y5=
\Y6=
\Y7=
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TI-84 Plus

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Graph Func  :Y=
Y1■1.5ln (X²+1) [—]
Y2■Y1+2         [—]
Y3■Y1-2         [—]
Y4:             [—]
Y5:             [—]
Y6:             [—]
SEL DEL TYPE STYL MEM DRAW
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Casio fx-9860G

5. Topic: Integration

(i) $\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9x^2}\,\mathrm{d}x = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+(3x)^2}\,\mathrm{d}x$

$= \frac{1}{3}\int_0^{\frac{1}{\sqrt{3}}} \frac{3}{1+(3x)^2}\,\mathrm{d}x$

Express in $\frac{\mathrm{f}'(x)}{1+[\mathrm{f}(x)]^2}$

$= \frac{1}{3}\left[\tan^{-1}(3x)\right]_0^{\frac{1}{\sqrt{3}}}$

$$\int \frac{\mathrm{f}'(x)}{1^2+[\mathrm{f}(x)]^2}\,\mathrm{d}x = \tan^{-1}[\mathrm{f}(x)] + c$$

$= \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1}(0)\right]$

$= \frac{1}{3}\left[\frac{\pi}{3} - 0\right]$

$= \boldsymbol{\frac{\pi}{9}}$

Integration by parts:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

(ii) $\int_1^{\mathrm{e}} x^n \ln x\,\mathrm{d}x = \left[\ln x \cdot \left(\frac{x^{n+1}}{n+1}\right)\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \left(\frac{x^{n+1}}{n+1}\right)\left(\frac{1}{x}\right)\mathrm{d}x$

$= \left[\ln \mathrm{e} \cdot \left(\frac{\mathrm{e}^{n+1}}{n+1}\right) - \ln 1 \cdot \left(\frac{1^{n+1}}{n+1}\right)\right] - \int_1^{\mathrm{e}} \left(\frac{x^n}{n+1}\right)\mathrm{d}x$

$= \frac{\mathrm{e}^{n+1}}{n+1} - \frac{1}{n+1}\left[\frac{x^{n+1}}{n+1}\right]_1^{\mathrm{e}}$

$= \frac{\mathrm{e}^{n+1}}{n+1} - \frac{1}{(n+1)^2}\left(\mathrm{e}^{n+1} - 1^{n+1}\right)$

$= \frac{1}{(n+1)^2}\left[(n+1)\mathrm{e}^{n+1} - \mathrm{e}^{n+1} + 1\right]$

$= \boldsymbol{\frac{n\mathrm{e}^{n+1}+1}{(n+1)^2}}$

ILATE/LIATE Rule:

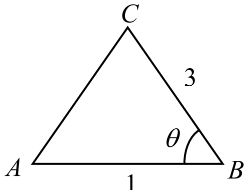
Sub $u = \ln x$ (L)

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

Sub $\frac{\mathrm{d}v}{\mathrm{d}x} = x^n$(A)

$v = \frac{x^{n+1}}{n+1}$

6. Topic: Maclaurin's Series



(a) By Cosine Rule, $\cos\angle ABC = \frac{AB^2 + BC^2 - AC^2}{2\,(AB)(BC)}$

$\cos\theta = \frac{1^2 + 3^2 - AC^2}{2\,(1)(3)}$

$= \frac{10 - AC^2}{6}$

$\Rightarrow AC^2 = 10 - 6\cos\theta$

$\approx 10 - 6\left(1 - \frac{1}{2}\theta^2\right)$

$\approx 10 - 6 + 3\theta^2$

$\approx 4 + 3\theta^2$

$\therefore AC \approx \left(4 + 3\theta^2\right)^{\frac{1}{2}}$ **(Shown)**

Small Angle Approximation:

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2$$

$\approx 4^{\frac{1}{2}}\underbrace{\left(1 + \frac{3}{4}\theta^2\right)^{\frac{1}{2}}}$

Express in $(1+x)^n$ since n is not a positive integer

$\approx 2\left[1 + \frac{1}{2}\left(\frac{3}{4}\theta^2\right) + \ldots\right]$

Maclaurin's expansion:

$(1+x)^n = 1 + nx + \cdots$

x^2 term i.e. $\left(\frac{3}{4}\theta^2\right)^2$ & above ignored since θ is small

$\approx 2\left[1 + \frac{3}{8}\theta^2 + \ldots\right]$

$\approx 2 + \frac{3}{4}\theta^2$

$\therefore \boldsymbol{a = 2,\ b = \frac{3}{4}}$

(b) $\mathrm{f}(x) = \tan\left(2x + \frac{1}{4}\pi\right)$ $\Rightarrow \mathrm{f}(0) = \tan\left(\frac{1}{4}\pi\right) = \mathbf{1}$

$\mathrm{f}'(x) = 2\sec^2\left(2x + \frac{1}{4}\pi\right)$ $\Rightarrow \mathrm{f}'(0) = 2\sec^2\left(\frac{1}{4}\pi\right) = \mathbf{4}$

$\mathrm{f}''(x) = 2 \cdot 2\sec\left(2x + \frac{1}{4}\pi\right)\left[2\sec\left(2x + \frac{1}{4}\pi\right)\tan\left(2x + \frac{1}{4}\pi\right)\right]$

$= 8\sec^2\left(2x + \frac{1}{4}\pi\right)\tan\left(2x + \frac{1}{4}\pi\right)$

$\Rightarrow \mathrm{f}''(0) = 8(2)(1) = \mathbf{16}$

$\frac{\mathrm{d}}{\mathrm{d}x}\sec^n[\mathrm{f}(x)]$

$= n\sec^{n-1}[\mathrm{f}(x)] \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sec[\mathrm{f}(x)]$

$\frac{\mathrm{d}}{\mathrm{d}x}\sec[\mathrm{f}(x)]$

$= \mathrm{f}'(x) \cdot \sec[\mathrm{f}(x)]\tan[\mathrm{f}(x)]$

$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots$

$= 1 + x(4) + \left(\frac{x^2}{2!}\right)16 + \ldots$

$= \mathbf{1 + 4}\boldsymbol{x}\mathbf{ + 8}\boldsymbol{x^2}\mathbf{ + \ldots}$

7. **Topic: Differentiation**

Total time taken: $180 = (2y + x)(3) + \pi\left(\frac{x}{2}\right)(9)$

$$60 = 2y + x + \pi\left(\frac{x}{2}\right)(3)$$

$$2y = 60 - x - \frac{3\pi x}{2}$$

$$y = 30 - \frac{x}{2} - \frac{3\pi x}{4} \text{ ……… (1)}$$

Total time taken =
length of straight part × 3 hrs/m +
length of semicircular part × 9 hrs/m

Total area of rectangular & semicircular parts:

$$A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$= x\underbrace{\left(30 - \frac{x}{2} - \frac{3\pi x}{4}\right)}_{} + \frac{1}{2}\pi\left(\frac{x^2}{4}\right)$$

$$= 30x - \frac{x^2}{2} - \frac{3\pi x^2}{4} + \frac{\pi x^2}{8}$$

$$= 30x - \frac{x^2}{2} - \frac{5\pi x^2}{8}$$

Sub y expression
from (1)

When A is maximum, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0.$

$$\Rightarrow 30 - x - \frac{10\pi x}{8} = 0$$

$$x + \frac{10\pi x}{8} = 30$$

$$x = \frac{30}{1 + \frac{10\pi}{8}}$$

$$\approx 6.0889$$

$$\approx \mathbf{6.09\ (3\ sig.\ fig.)}$$

Second derivative test: $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -1 - \frac{5\pi}{4} < 0 \Rightarrow A$ is maximum when $x = 6.09$

Sub $x \approx 6.0889$ into (1): $y = 30 - \frac{6.0889}{2} - \frac{3\pi(6.0889)}{4}$

$$\approx 12.608$$

$$\approx \mathbf{12.6\ (3\ sig.\ fig.)}$$

∴ x = 6.09 & y = 12.6 gives the flower-bed a maximum area.

8. **Topic: Complex Numbers**

(i) $z_1{}^3 = \left(1 + \sqrt{3}\mathrm{i}\right)^3$

$$= 1 + 3\sqrt{3}\mathrm{i} + 3\left(\sqrt{3}\mathrm{i}\right)^2 + \left(\sqrt{3}\mathrm{i}\right)^3$$

$$= 1 + 3\sqrt{3}\mathrm{i} - 3(3) - 3\sqrt{3}\mathrm{i}$$

$$= \mathbf{-8}$$

Binomial Expansion:
$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$

Alternative Method
$z = 1 + \sqrt{3}\mathrm{i} = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$
$z^3 = \left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)^3 = 2^3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} = -8$

(ii) Given that $1 + \sqrt{3}\mathrm{i}$ is a root, sub $z = 1 + \sqrt{3}\mathrm{i}$ into $2z^3 + az^2 + bz + 4 = 0$:

$$2(-8) + a\left(1 + \sqrt{3}\mathrm{i}\right)^2 + b\left(1 + \sqrt{3}\mathrm{i}\right) + 4 = 0$$

$$-16 + a\left(1 + 2\sqrt{3}\mathrm{i} - 3\right) + b + b\sqrt{3}\mathrm{i} + 4 = 0$$

$$-16 + 2a\sqrt{3}\mathrm{i} - 2a + b + b\sqrt{3}\mathrm{i} + 4 = 0$$

$$-12 - 2a + b + 2a\sqrt{3}\mathrm{i} + b\sqrt{3}\mathrm{i} = 0$$

$z^3 = -8$
from

Comparing real parts:

$$-12 - 2a + b = 0$$

$$-2a + b = 12$$

$$b = 12 + 2a \text{ … … … (1)}$$

Comparing imaginary parts:

$$2a\sqrt{3} + b\sqrt{3} = 0$$

$$b = -2a \text{ … … … . … (2)}$$

Equating (1) & (2):

$$12 + 2a = -2a$$

$$4a = -12$$

$$\Rightarrow a = \mathbf{-3}$$

$$b = -2(-3) = \mathbf{6}$$

(iii) Since the equation $2z^3 - 3z^2 + 6z + 4 = 0$ has real coefficients,

Given $1 + \sqrt{3}\mathrm{i}$ is a root $\Rightarrow 1 - \sqrt{3}\mathrm{i}$ is also a root.

$\Rightarrow$ A quadratic factor of the equation is

$$\left[z - \left(1 + \sqrt{3}\mathrm{i}\right)\right]\left[z - \left(1 - \sqrt{3}\mathrm{i}\right)\right] = z^2 - 2z + 4$$

Non-real roots occur in conjugate pairs in polynomial equations with real coefficients.

$$\Rightarrow 2z^3 - 3z^2 + 6z + 4 \equiv (z^2 - 2z + 4)(2z + 1) = 0$$

$$\Rightarrow \boldsymbol{z = 1 + \sqrt{3}\mathrm{i}, 1 - \sqrt{3}\mathrm{i}, -\frac{1}{2}}$$

By inspection or long division.

ALTERNATE APPROACH

(ii) Since the equation $2z^3 + az^2 + bz + 4 = 0$ has real coefficients,
Given $1 + \sqrt{3}i$ is a root $\Rightarrow 1 - \sqrt{3}i$ is also a root.

$\Rightarrow$ A quadratic factor of the equation is

$$[z - (1 + \sqrt{3}i)][z - (1 - \sqrt{3}i)] = z^2 - 2z + 4$$

Non-real roots occur in conjugate pairs in polynomial equations with real coefficients.

$\Rightarrow$ $2z^3 + az^2 + bz + 4 \equiv (z^2 - 2z + 4)(Az - B)$ where $A, B \in \mathbb{R}$

Comparing coefficients of z^3, $A = 2$

Comparing coefficients of constant, $4 = -4B \Rightarrow B = -1$

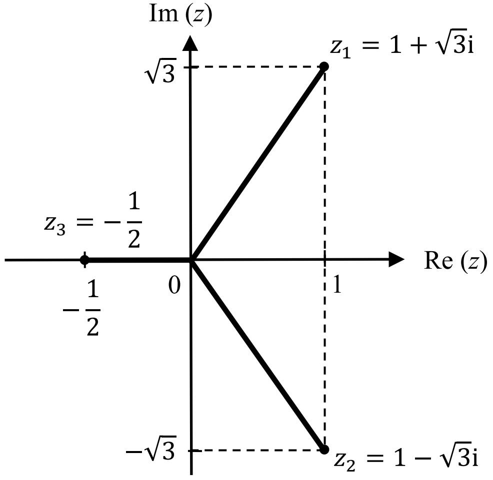
Comparing coefficients of z^2, $a = -B - 2A = -(-1) - 2(2) = \mathbf{-3}$

Comparing coefficients of z, $b = -2(-B) + 4A = -2[-(-1)] + 4(2) = \mathbf{6}$

(iii) Sub $a = -3$, $b = 6$, $A = 2$, $B = -1$ in (ii),

$$2z^3 - 3z^2 + 6z + 4 \equiv [z - (1 + \sqrt{3}i)][z - (1 - \sqrt{3}i)](2z + 1) = 0$$

$$\Rightarrow \boldsymbol{z = 1 + \sqrt{3}i, 1 - \sqrt{3}i, -\frac{1}{2}}$$



9. **Topic: Graphing Techniques**

(i) $f(x) = \frac{ax+b}{cx+d}$

$$f'(x) = \frac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2}$$

$$= \frac{acx + ad - acx - bc}{(cx + d)^2}$$

$$= \frac{ad - bc}{(cx + d)^2} \neq 0, \forall x \because ad - bc \neq 0 \text{ (Given)}$$

Quotient Rule:
$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$\therefore$ **By differentiation, $f'(x) \neq 0$, $\forall x$**
$\Rightarrow$ **the graph of $y = f(x)$ has no turning points.**

(ii) $f(x) = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d}$

$$= \frac{a}{c} + \frac{bc - ad}{c\,(cx+d)}$$

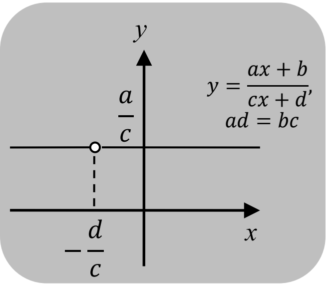
$$= \frac{a}{c} - \frac{ad - bc}{c(cx+d)}$$

By long division:

$$\begin{array}{r|l} & \frac{a}{c} \\ \hline cx + d & ax \quad + \quad b \\ & ax \quad + \quad \frac{ad}{c} \\ \hline & \qquad b - \frac{ad}{c} \end{array}$$

When $ad - bc = 0$, $y = f(x) = \frac{a}{c}, x \neq -\frac{d}{c}$

$\therefore$ **The graph is a horizontal line at $y = \frac{a}{c}$, but undefined at $x = -\frac{d}{c}$**



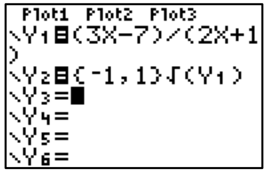
(iii) $y = \frac{3x-7}{2x+1}$

Sub $a = 3$, $b = -7$, $c = 2$, $d = 1$ into $f'(x)$ in (i):

$$\Rightarrow \frac{dy}{dx} = \frac{3(1) - (-7)(2)}{(2x+1)^2} = \frac{17}{(2x+1)^2} > 0$$

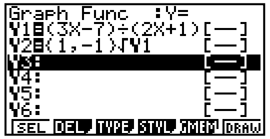
$\therefore$ **the graph has a positive gradient at all points since $(2x + 1)^2 > 0$, $x \neq -\frac{1}{2}$.**

(iv)



```
 Plot1  Plot2  Plot3
\Y1■(3X-7)/(2X+1
)
\Y2■{-1,1}√(Y1)
\Y3=■
\Y4=
\Y5=
\Y6=
```

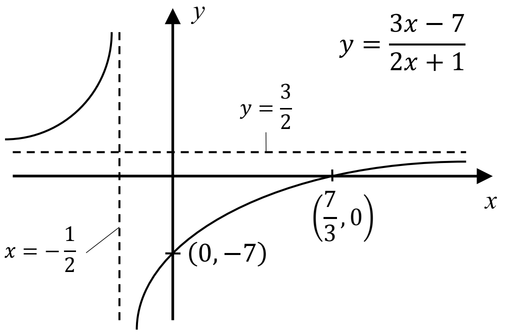
TI-84 Plus

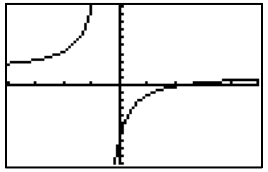


```
Graph Func   :Y=
Y1■(3X-7)÷(2X+1)[—]
Y2■{1,-1}√Y1    [—]
Y3:             [—]
Y4:             [—]
Y5:             [—]
Y6:             [—]
```

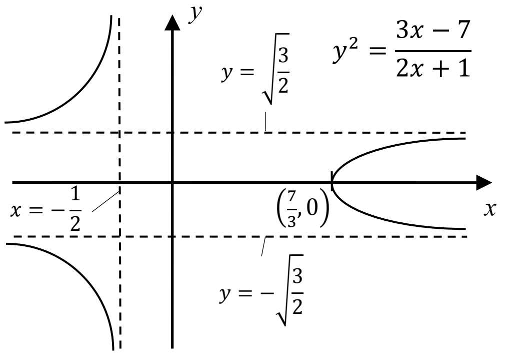
Casio fx-9860G

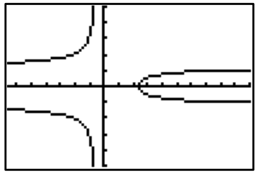
(a)





(b)





10. **Topic: Arithmetic & Geometric Series**

(i) Since the student saves \$3 more than the previous month in each subsequent month, the monthly savings = 10, 13, 16, 19, …
⇒ Arithmetic series with the 1st term, $a = 10$, and common difference, $d = 3$.

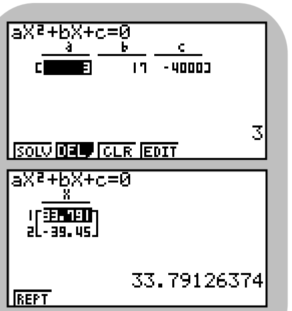
To save over \$2000 in total, $S_n > 2000$

$$\frac{n}{2}[2a + (n-1)d] > 2000$$

Sub $a = 10$, $d = 3$ into S_n:

$$\frac{n}{2}[2(10) + (n-1)(3)] > 2000$$
$$\frac{n}{2}[17 + 3n] > 2000$$
$$17n + 3n^2 > 4000$$
$$3n^2 + 17n - 4000 > 0$$

Sum of A. P.: $S_n = \frac{n}{2}[2a + (n-1)d]$



```
aX²+bX+c=0
    a       b      c
[      3    17  -4000]

                     3
```

```
aX²+bX+c=0
    X
1[33.791]
2[-39.45]

           33.79126374
```

Casio fx-9860G

$n > 33.79$ or $n < -39.46$ (reject)
⇒ $n = 34$ (earliest no. of months since 1 Jan 2009)

∴ she will first have saved over \$2000 on the 1st of October 2011.

(ii) (a) Taking into account only the original \$10 deposit:

Month	Balance at Start of Month (\$)	Interest Earned End of Month (\$)
1	10	(0.02)(10)
2	10+(0.02)(10) = (1.02)(10)	(0.02)(1.02)(10)
3	(1.02)(10)+(0.02)(1.02)(10) = (1.02)(1.02)(10)	(0.02)(1.02)(1.02)(10)
n	$(1.02)^{n-1}(10)$	$(0.02)(10)(1.02)^{n-1}$

Total compound interest earned after n months from the original \$10

$= (10)(0.02) + (10)(0.02)[1.02 + 1.02^2 + \ldots + 1.02^{n-1}]$

$= (10)(0.02) + (10)(0.02)\left[\frac{1.02(1.02^{n-1}-1)}{1.02-1}\right]$

$= (10)(0.02)\left[1 + \frac{1.02^n - 1.02}{0.02}\right]$

$= 10(1.02^n - 1)$

Sum of G.P. where $r > |1|$: $S_n = \frac{a(r^n - 1)}{r - 1}$

$\therefore$ total interest earned in 2 years (24 months) from original \$10

$= 10(1.02^{24} - 1)$

$\approx$ **\$6.08 (3 sig. fig.)**

(b) Account balance including \$10 deposited monthly:

Month	Balance at Start of Month (\$)	Balance at End of Month (\$)
1	10	(1.02)(10)
2	(1.02)(10) + 10	1.02[1.02(10) + 10] $= 1.02^2(10) + 1.02(10)$
3	$1.02^2(10) + 1.02(10) + 10$	$1.02[1.02^2(10) + 1.02(10) + 10]$ $= 1.02^3(10) + 1.02^2(10) + 1.02(10)$
n		$(1.02 + 1.02^2 + \ldots + 1.02^n)10$ $= 10\left[\frac{1.02(1.02^n - 1)}{1.02 - 1}\right]$ $= 510(1.02^n - 1)$ ……… (1)

Total in account at end of 2 years (24 months)

$= 510(1.02^{24} - 1)$

Sub $n = 24$ into (1)

≈ 310.3029972

$\approx$ **\$310 (3 sig. fig.)**

(c) Let n be the number of complete months taken for the total in the account to first exceed \$2000.

Using (1) from (b)(ii),

$$\begin{aligned} 510(1.02^n - 1) &> 2000 \\ 1.02^n &> 4.9215 \\ n \ln 1.02 &> \ln 4.9215 \\ n &> 80.475 \end{aligned}$$

$\therefore$ **n = 81 months.**

11. **Topic: Three-Dimensional Geometry**

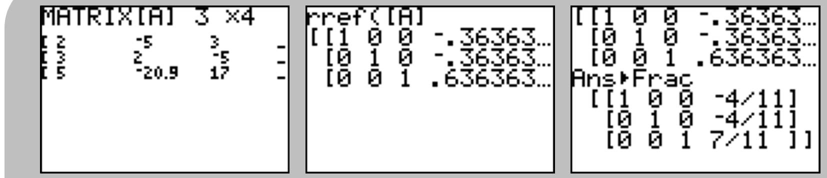
Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the point of intersection of p_1, p_2, p_3. Solving for p_1, p_2 and p_3,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\begin{array}{ccc|c} 2 & -5 & 3 & 3 \\ 3 & 2 & -5 & -5 \\ 5 & -20.9 & 17 & 16.6 \end{array}\right) = \begin{pmatrix} -\frac{4}{11} \\ -\frac{4}{11} \\ \frac{7}{11} \end{pmatrix}$$

Cartesian equation of plane

$\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d$:

$n_1 x + n_2 y + n_3 z = d$



```
MATRIX[A] 3 ×4
[ 2       ⁻5     3    _
[ 3       2      ⁻5   _
[ 5       ⁻20.9  17   _
```

```
rref([A]
[[1 0 0 ⁻.36363…
 [0 1 0 ⁻.36363…
 [0 0 1 .636363…
```

```
[[1 0 0 ⁻.36363…
 [0 1 0 ⁻.36363…
 [0 0 1 .636363…
Ans▸Frac
 [[1 0 0 ⁻4/11]
  [0 1 0 ⁻4/11]
  [0 0 1 7/11 ]]
```

TI-84 Plus

(i) Since l lies on p_1 and p_2 $\Rightarrow$ solve for p_1 and p_2,

$$2x - 5y + 3z = 3 \ldots\ldots\ldots\ldots (1)$$
$$3x + 2y - 5z = -5 \ldots\ldots\ldots (2)$$

$(1) \times 2 + (2) \times 5$:
$19x - 19z = -19 \Rightarrow z = x + 1$

Eliminate y

$(1) \times 3 - (2) \times 2$:
$-19y + 19z = 19 \Rightarrow z = y + 1$

Eliminate x

$$\Rightarrow \frac{x+1}{1} = \frac{y+1}{1} = \frac{z}{1}$$

Cartesian equation of line $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + s\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

$$\frac{x - a_1}{b_1} = \frac{y + a_2}{b_2} = \frac{z - a_3}{b_3}$$

$\therefore$ **vector equation of l is:** $\mathbf{r} = \begin{pmatrix} \mathbf{-1} \\ \mathbf{-1} \\ \mathbf{0} \end{pmatrix} + \boldsymbol{s}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}, \boldsymbol{s} \in \mathbb{R}$

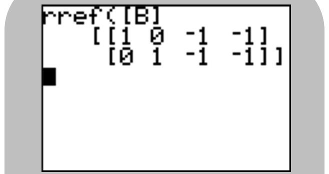
ALTERNATE APPROACH

Direction vector of l, $\widehat{\mathbf{d}}_l = \mathbf{n}_{p_1} \times \mathbf{n}_{p_2}$ (where $\mathbf{n}_{p_1}$ and $\mathbf{n}_{p_2}$ are the normal vectors of p_1 and p_2 respectively)

$$= \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} = 19\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Using the common point obtained earlier,

vector equation of l: $\mathbf{r} = \begin{pmatrix} -\frac{\mathbf{4}}{\mathbf{11}} \\ -\frac{\mathbf{4}}{\mathbf{11}} \\ \frac{\mathbf{7}}{\mathbf{11}} \end{pmatrix} + \boldsymbol{s}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}, \boldsymbol{s} \in \mathbb{R}$



```
rref([B]
   [[1 0  -1  -1]
    [0 1  -1  -1]]
```

TI-84 Plus

$\Rightarrow x - z = -1$
$\Rightarrow y - z = -1$

$$\mathbf{d}_l = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 3 \\ 3 & 2 & -5 \end{vmatrix}$$
$$= \begin{vmatrix} -5 & 3 \\ 2 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -5 \\ 3 & 2 \end{vmatrix}\mathbf{k}$$
$$= 19\mathbf{i} + 19\mathbf{j} + 19\mathbf{k}$$
$$\Rightarrow \widehat{\mathbf{d}}_l = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(ii) Given that l lies on $p_3 \Rightarrow$ all points on l lies on p_3.

Picking two points on l from the vector equation of l in (i),

Let $s = 0$: $\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ \lambda \\ 17 \end{pmatrix} = \mu$

$$-5 - \lambda = \mu \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $s = 1$: $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ \lambda \\ 17 \end{pmatrix} = \mu$

$$\mu = 17 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Sub (2) into (1): $-5 - \lambda = 17 \Rightarrow \lambda = -22$

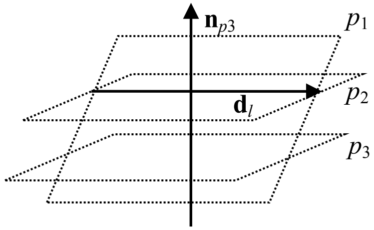
$\therefore$ **$\lambda = -22$, $\mu = 17$**

(iii) For the three planes to have no point in common,

a. p_3 must be parallel to l

$\Rightarrow \mathbf{n}_{p_3} \perp \hat{\mathbf{d}}_l$ direction vector of l

$$\begin{pmatrix} 5 \\ \lambda \\ 17 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$
$$5 + \lambda + 17 = 0$$
$$\lambda = -22$$



$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$

b. p_3 must not contain l

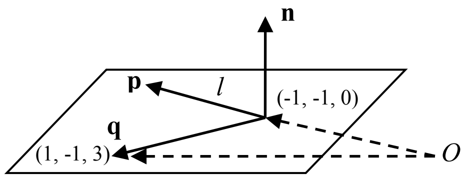
$\Rightarrow \mathbf{a}_l \cdot \mathbf{n}_{p_3} \neq \mu$

$$\Rightarrow \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -22 \\ 17 \end{pmatrix} \neq \mu$$
$$\mu \neq 17$$

$\mathbf{a}_l$: Pick a point on l obtained from (i)

$\therefore$ **Given that the three planes have no point in common, $\lambda = -22$ and μ can be any real number but not 17.**

(iv) Let **p** and **q** be two distinct direction vectors that lie on the plane (required to get the equation of the plane).



Since plane contains $l \Rightarrow \mathbf{p} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

Direction vector of l, $\hat{\mathbf{d}}_l$ obtained in (i).

Since plane contains the point (1, –1, 3) $\Rightarrow \mathbf{q} = \begin{pmatrix}1\\-1\\3\end{pmatrix} - \mathbf{a}_l$

Position vector of point on l, $\mathbf{a}_l$ obtained in (i).

$$= \begin{pmatrix}1\\-1\\3\end{pmatrix} - \begin{pmatrix}-1\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\3\end{pmatrix}$$

Normal vector to the plane, $\mathbf{n} = \mathbf{p} \times \mathbf{q} = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}2\\0\\3\end{pmatrix} = \begin{pmatrix}3\\-1\\-2\end{pmatrix}$

Vector equation of plane: $\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix}1\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\-2\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\-2\end{pmatrix} = -2$$

∴ **Cartesian equation of plane: $\mathbf{3x - y - 2z = -2}$**

Cartesian equation of plane

$\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = d$:

$$n_1x + n_2y + n_3z = d$$