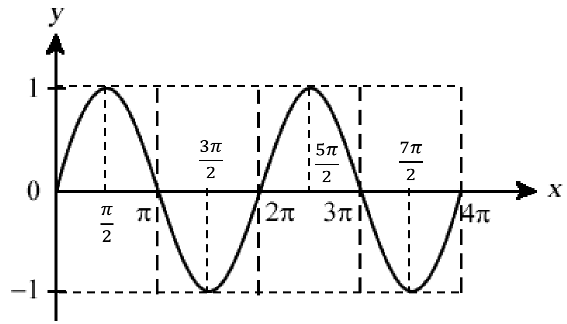
MATHEMATICS (H1) 8863/01
Paper 1 Suggested Solutions October/November 2008

1. **Topic: Graphs**

$y = \sin x,\ 0 \le x \le 4\pi$

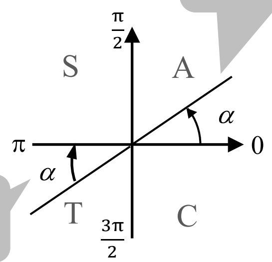


$\sin \alpha = c$

(i) $\sin(2\pi + \alpha) = \sin \alpha$
$= \boldsymbol{c}$

(ii) $\sin(3\pi + \alpha) = -\sin \alpha$
$= \boldsymbol{-c}$

2π + x will end up at 1st quad



3π + x will end up at 3rd quad

$$\sin x = -c$$
$$x = \boldsymbol{\pi + \alpha \text{ or } 2\pi - \alpha}$$

2. **Topic: Simultaneous Equations**

$$x + y = 20$$
$$x = 20 - y \text{ ………………… (1)}$$
$$x^2 + y^2 = 300 \text{ ………………………… (2)}$$

Sub (1) into (2),

$$(20 - y)^2 + y^2 = 300$$
$$400 - 40y + y^2 + y^2 = 300$$
$$2y^2 - 40y + 100 = 0$$
$$y^2 - 20y + 50 = 0$$
$$y = \frac{20 \pm \sqrt{20^2 - 4(1)(50)}}{2}$$
$$y = \frac{20 \pm \sqrt{200}}{2}$$
$$y = 10 + 5\sqrt{2} \text{ (rejected) or } 10 - 5\sqrt{2}$$

$$\therefore x = 20 - (10 - 5\sqrt{2})$$
$$= \boldsymbol{10 + 5\sqrt{2}}$$

3. **Topic: Simultaneous Equations, Integration**

$$y = 2x^2 \text{ …………………… (1)}$$
$$y = x^2 + k^2 \text{ ……………….. (2)}$$

(1) = (2),

$$2x^2 = x^2 + k^2$$
$$x^2 = k^2$$
$$x = k \text{ or } -k$$

$\therefore$ x-coordinates of P & Q are k or $-k$ **(Shown)**

Graphs are symmetrical about the y-axis.

Shaded area between c_1 & c_2 $= 2\int_0^k (x^2 + k^2 - 2x^2)\,dx$

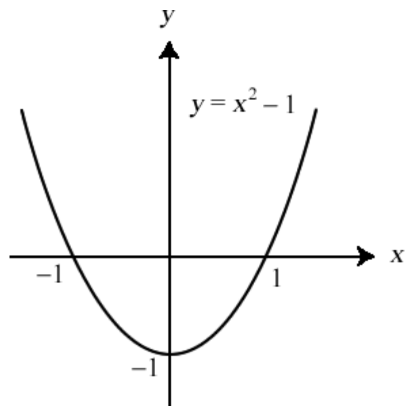
$$= 2\int_0^k (k^2 - x^2)\,dx$$
$$= 2\left[k^2 x - \frac{x^3}{3}\right]_0^k$$

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

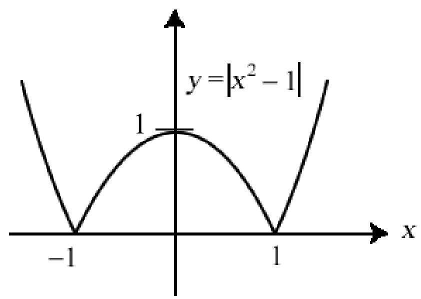
$$= 2\left[k^3 - \frac{k^3}{3}\right]$$
$$= 2\left[\frac{2}{3}k^3\right]$$
$$= \boldsymbol{\frac{4}{3}k^3}$$

4. **Topics: Functions**

(i) (a) $y = \mathrm{f}(x) = x^2 - 1$



(b) $y = \mathrm{gf}(x) = \mathrm{g}(x^2 - 1) = |x^2 - 1|$



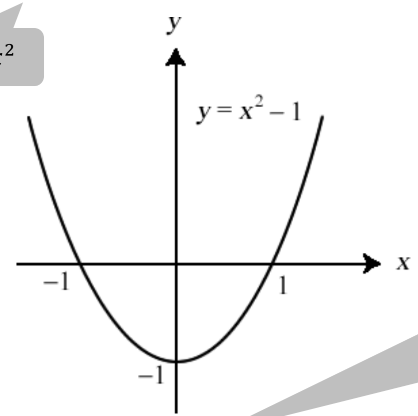
(c) $y = \mathrm{fg}(x)$

$= \mathrm{f}(|x|)$

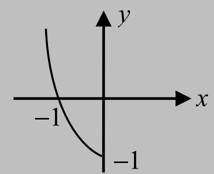
$= (|x|)^2 - 1$

$= x^2 - 1$

$(|x|)^2 = x^2$



f has to be a one-one function since $x \le a$.



(ii) For f to have an inverse $\Rightarrow$ Greatest value of $a = 0$

$D_{f^{-1}} = R_f = \mathbf{[-1, \infty]}$

Let $y = x^2 - 1$

$y + 1 = x^2$

$x = \pm\sqrt{y+1}$

$\therefore \mathbf{f^{-1}(x) = -\sqrt{x+1}}$

Take the –ve part of the graph since $x \le a$

5. **Topic: Differentiation**

$$x = t^3 - 12t^2 + kt$$

$$\frac{dx}{dt} = 3t^2 - 24t + k$$

Since x is an increasing function of t

$$\Rightarrow \quad \frac{dx}{dt} > 0$$

$$(3t^2 - 24t + k) > 0$$

$$\Rightarrow \quad b^2 - 4ac < 0$$

> $ax^2 + bx + c > 0$ for all real values of x
> $\Leftrightarrow b^2 - 4ac < 0$ and $a > 0$

$$(-24)^2 - 4(3)(k) < 0$$

$$576 - 12k < 0$$

$$576 < 12k$$

$$\boldsymbol{k > 48}$$

(i) $x = t^3 - 12t^2 + 36t$
$= t(t^2 - 12t + 36)$
$= t(t-6)^2$

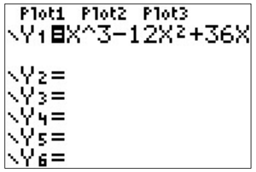
$$\frac{dx}{dt} = 3t^2 - 24t + 36$$

When $\frac{dx}{dt} = 0$

$$t^2 - 8t + 12 = 0$$

$$(t-6)(t-2) = 0$$

$t = 2$ or $t = 6$



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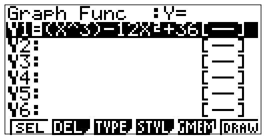

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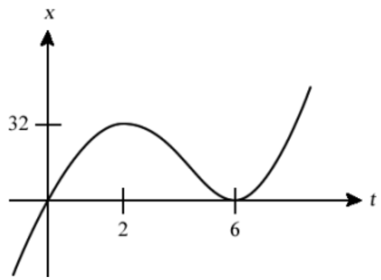
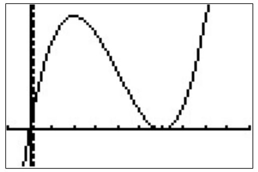


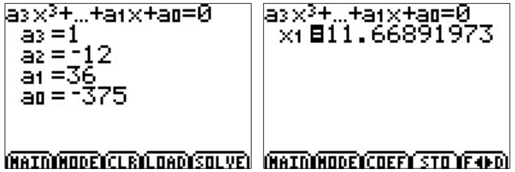


(ii) $t^3 - 12t^2 + 36t = 375$

$$t^3 - 12t^2 + 36t - 375 = 0$$

$$t = 11.66892$$

$$t \approx \textbf{11.7 (3 sig. fig.)}$$



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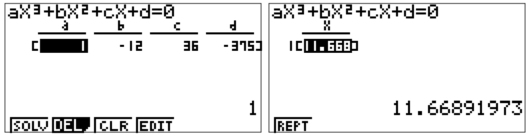

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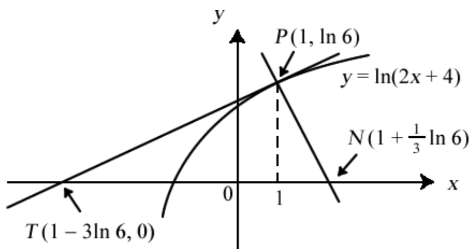
6. **Topic: Applications of Differentiation**

$$y = \ln(2x+4)$$
$$\frac{dy}{dx} = \frac{2}{2x+4} = \frac{1}{x+2}$$

When $x = 1$, $\frac{dy}{dx} = \frac{1}{3}$

Equation of tangent at P:

$$\Rightarrow \; y - \ln 6 = \frac{1}{3}(x-1)$$
$$y = \frac{1}{3}x - \frac{1}{3} + \ln 6$$



When $y = 0$,

$$\frac{1}{3}x - \frac{1}{3} + \ln 6 = 0$$
$$x - 1 + 3\ln 6 = 0$$
$$x = 1 - 3\ln 6$$

$\therefore x$-coordinate of $T = 1 - 3\ln 6$ **(Shown)**

Gradient of $PN = -3$

Gradient of $TP = \frac{1}{3}$

$\Rightarrow$ Gradient $PN = -\frac{1}{\frac{1}{3}} = -3$ ($TP \perp PN$)

Equation of PN:

$$\Rightarrow \; y - \ln 6 = -3(x-1)$$
$$y = -3x + 3 + \ln 6$$

When $y = 0$,

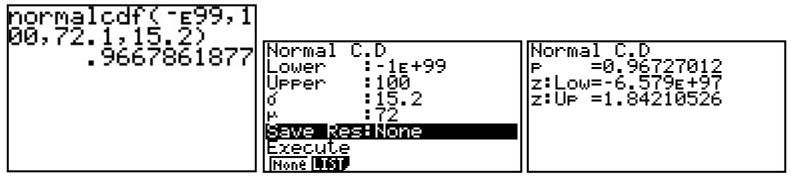
$$-3x + 3 + \ln 6 = 0$$
$$3x = 3 + \ln 6$$
$$x = 1 + \frac{1}{3}\ln 6$$

$\therefore x$-coordinate of $N =$ $\mathbf{1 + \frac{1}{3}\ln 6}$

$$\text{Area of } \Delta PTN = \frac{1}{2}(\ln 6)\left[1 + \frac{1}{3}\ln 6 - (1 - 3\ln 6)\right]$$
$$= \frac{1}{2}(\ln 6)\left[\frac{1}{3}\ln 6 + 3\ln 6\right]$$
$$= \mathbf{\frac{5}{3}(\ln 6)^2 \text{ units}^2}$$

7. **Topic: Normal Distribution**

Using this distribution, we have $P(X > 100) = 1 - 0.967 = 0.033$ which is impossible since the maximum marks is 100. Hence, normal distribution is not a good approximation.



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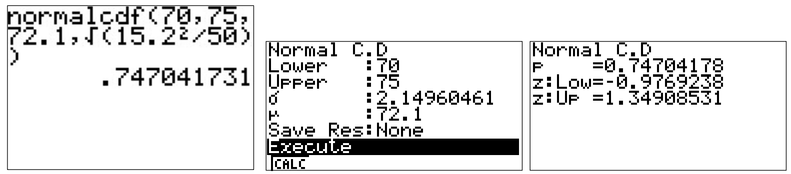
Let random variable X be the mean mark for all candidates,

i.e. $X \sim N(72.1, 15.2^2)$

For a random sample of 50 candidates, by Central Limit Theorem:

$$\bar{X} \sim N\left(72.1, \frac{15.2^2}{50}\right)$$

$P(70 < \bar{X} < 75) =$ **0.747 (3 sig. fig.)**



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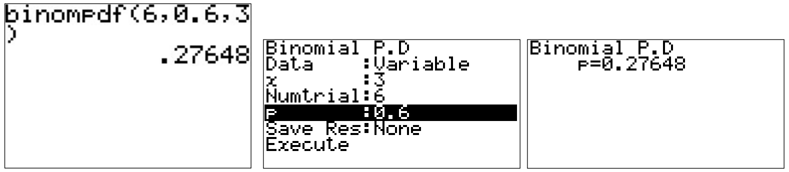
8. **Topic: Normal Distribution, Hypothesis Testing**

Let the random variable X be the number of loaves of bread, out of 6 loaves, which are crusty.

$X \sim B(6, 0.6)$

$P(X = 3) = 0.2764$

$\approx$ **0.276 (3 sig. fig.)**



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Let the random variable Y be the number of loaves of bread, out of 40 loaves, which are crusty.

$n = 40, p = 0.6, q = 0.4$

Since n is large, $np = 24 > 5$ and $nq = 16 > 5$

$\therefore Y \sim B(40, 0.6)$

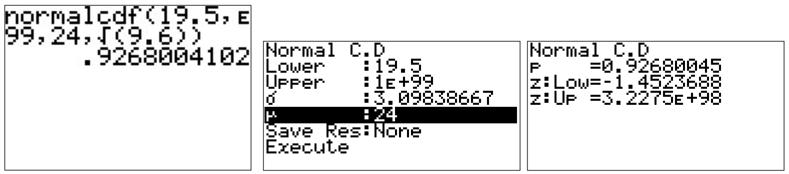
$\sim N(40 \times 0.6, 40 \times 0.6 \times 0.4)$ approximately

$\sim N(24, 9.6)$

$P(Y \geq 20) = P(Y > 19.5)$ [Continuity Correction]

$=$ **0.927**

Use continuity correction to approximate a discrete distribution (i.e binomial) by a continuous distribution (i.e normal).



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Let the random variable M be the mass of a loaf.

$M \sim N(1.24, \sigma^2)$

$P(M < 1) = 0.04$

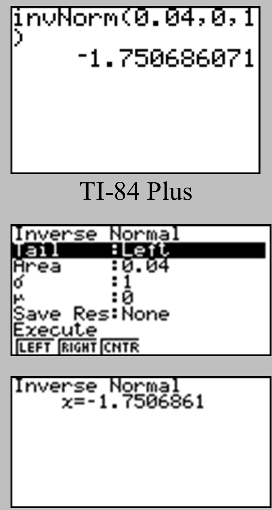
$P\left(Z < \frac{1-1.24}{\sigma}\right) = 0.04$

$\frac{1-1.24}{\sigma} = -1.75068$

$-0.24 = -1.75068\sigma$

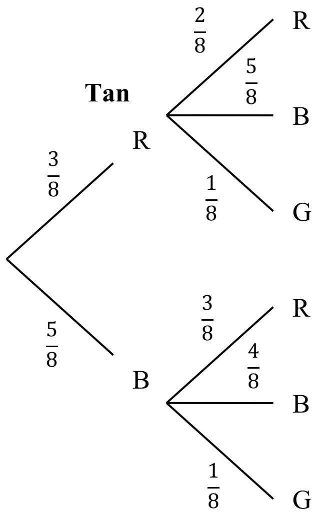
$\sigma = 0.13708$

$\boldsymbol{\sigma} \approx$ **0.137 (3 sig. fig.)**



9. **Topic: Probability**

(i)



(ii) P(Mui's pen is blue | Tan's pen is red) $= \dfrac{\text{P(Tan's pen red AND Mui's pen blue)}}{\text{P(Tan's pen red)}}$

$= \dfrac{\frac{3}{8}\left(\frac{5}{8}\right)}{\frac{3}{8}}$

$= \mathbf{\dfrac{5}{8}}$

(iii) P(Mui's pen is red) $= \frac{3}{8}\left(\frac{2}{8}\right) + \frac{5}{8}\left(\frac{3}{8}\right)$ RR + BR

$= \mathbf{\dfrac{21}{64}}$

(iv) P(Tan's pen is red | Mui's pen is blue) $= \dfrac{\text{P(Tan's pen red AND Mui's pen blue)}}{\text{P(Mui's pen blue)}}$

$= \dfrac{\frac{3}{8}\left(\frac{5}{8}\right)}{\frac{3}{8}\left(\frac{5}{8}\right)+\frac{5}{8}\left(\frac{4}{8}\right)}$

$= \mathbf{\dfrac{3}{7}}$

10. Topic: Hypothesis Testing

$\bar{x} = \dfrac{\sum x}{n}$

$= \dfrac{10317}{70}$

$= 147.38$

σ_x = unbiased estimate of population variance

s_x = sample variance

$\sigma_x^{\,2} = \dfrac{n}{n-1} s_x^{\,2}$

$= \dfrac{n}{n-1}\left[\dfrac{\sum x^2}{n} - (\bar{x})^2\right]$

$= \dfrac{70}{69}\left[\dfrac{1540231}{70} - (147.3857)^2\right]$

$= 284.824$

$\bar{X} \sim N\left(\mu, \dfrac{\sigma_x^{\,2}}{n}\right)$

$\sim N\left(150, \dfrac{284.824}{70}\right)$

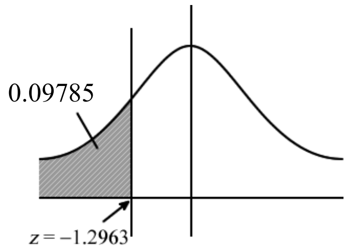
$H_0: \mu = 150$

$H_1: \mu < 150$

$z = \dfrac{\bar{x}-\mu}{\frac{\widehat{\sigma_x}}{\sqrt{n}}}$

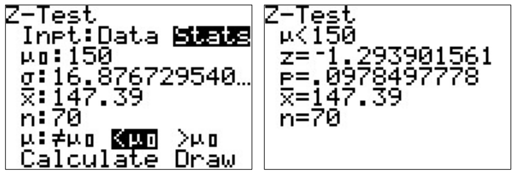
$= \dfrac{147.385-150}{\sqrt{\frac{284.824}{70}}}$

$= -1.2963$



From G.C., *p*-value = 0.09785

⇒ 9.7% of the sample batteries size of 70 has lifetime less than 150 hours



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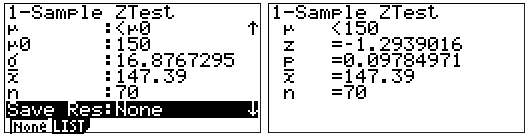

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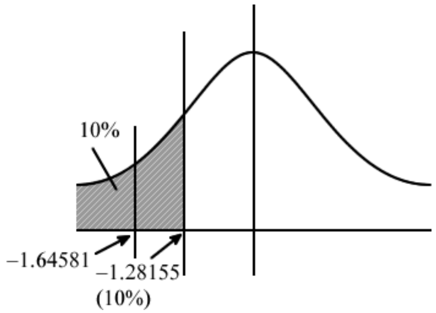
$$\text{Mean} = \frac{\sum x+\sum y}{n_1+n_2} = \frac{10317+7331}{70+50} = 147.067$$

$$\text{Unbiased estimate of population variance} = \frac{n}{n-1}\left[\frac{\sum x^2+\sum y^2}{n_1+n_2} - (\text{mean})^2\right] = \frac{120}{119}\left[\frac{1540231+1100565}{120} - 147.067^2\right] = 381.107$$

$H_0: \mu = 150$

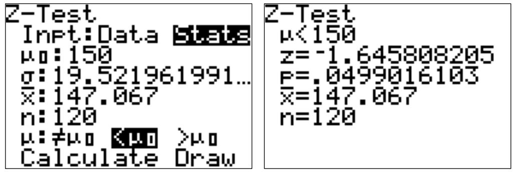
$H_1: \mu < 150$

$$z = \frac{147.067-150}{\sqrt{\frac{381.107}{120}}} = -1.64581$$



Since p-value = 4.99% < 10% , we reject H_o .

∴ there is sufficient evidence, at the 10% significance level that $\mu < 150$.



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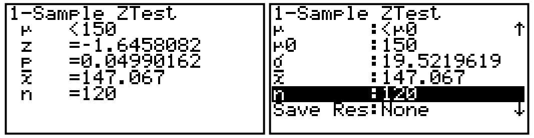

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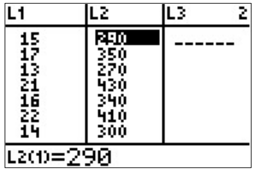
11. **Topic: Correlation Coefficient and Linear Regression**

(i)



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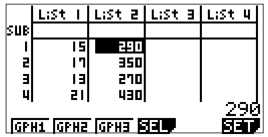

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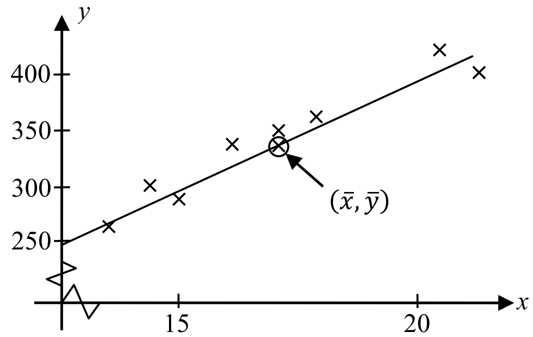
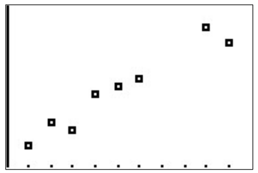
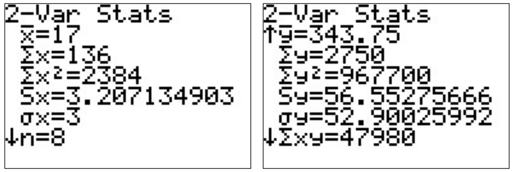




(ii) From G.C.,

$\bar{x} = \mathbf{17}$

$\bar{y} = \mathbf{343.75}$



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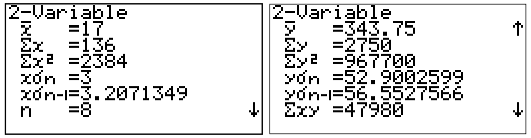

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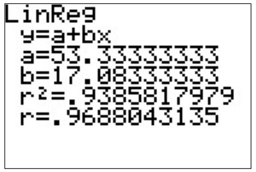
(iii) From G.C, $y = 53.3 + 17.1x$

(iv) From G.C.,

Product moment correlation coefficient, $\boldsymbol{r = 0.969}$
Since r is close to 1, this confirms a strong positive correlation between x and y.



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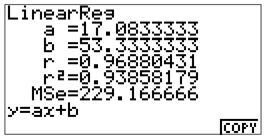

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(v) $x = 20$,
$y = 53.3 + 17.1(20)$
$= 395$
$\therefore$ Corresponding profit = **395 (in thousand dollars)**

(vi) As $x = 40$ falls outside the range $15 \le x \le 22$, we would be extrapolating and hence it would not give a reliable estimate of y.

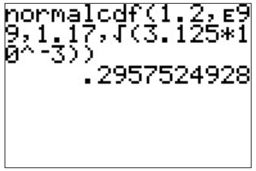
12. **Topics: Normal Distributions**

Let M = Mass of a apple
$M \sim N(0.234, 0.025^2)$

Let $S = M_1 + M_2 + M_3 + M_4 + M_5$
i.e. $S \sim N(1.17, 3.125 \times 10^{-3})$

Let $T = M_1 + M_2 + \ldots M_{10}$
i.e. $T \sim N(2.34, 6.25\times10^{-3})$

$P(M_1 + M_2 + \ldots M_5 > 1.2 \text{ kg}) = P(S > 1.2)$
$= 0.2957$
$\approx \mathbf{0.296 \text{ (3 sig. fig.)}}$



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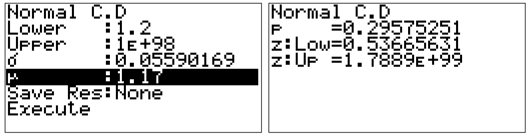

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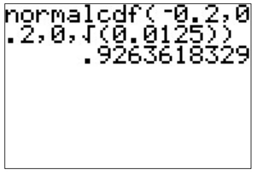
$(S_1 + S_2) - T \sim N[2(1.17) - 2.34, 2\times0.00315 + 0.00625]$
$\sim N(0, 0.0125)$

$P(|(S_1 + S_2) - T| \le 0.2) = P(-0.2 \le [S_1 + S_2 - T] \le 0.2)$
$= 0.9263$
$\approx \mathbf{0.926 \text{ (3 sig. fig.)}}$



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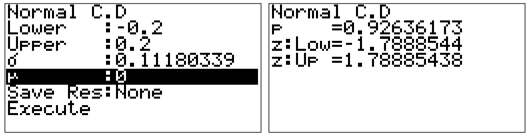

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$P[(S_1 + S_2)1.5 - 1.2(T) \geq 0.5\ ] = P\left[(S_1 + S_2) - 0.8T \geq \frac{1}{3}\right]$

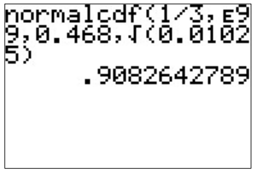
$(S_1 + S_2) - 0.8T$

$\sim N[\ 2 \times 1.17 - 0.8 \times 2.34,\ 2(3.125 \times 10^{-3}) + 0.8^2(6.25 \times 10^{-3})]$

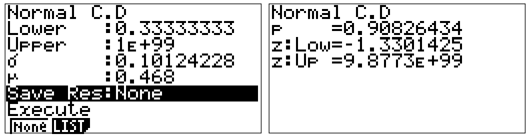
$\sim N(0.468,\ 0.01025)$

$\therefore P\left[(S_1 + S_2) - 0.8T \geq \frac{1}{3}\right] = 0.908264$

$\approx$ **0.908 (3 sig. fig.)**



TI-84 Plus



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Alternative Method:

Let Y be the random variable denoting the amount of money in $ that Lee pays more than Foo.

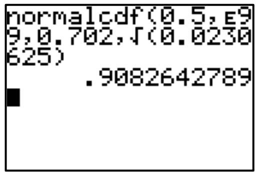
$$\begin{aligned} Y &= 1.5\,(S_1 + S_2) - 1.2\,T \\ E(Y) &= E[1.5\,(S_1 + S_2) - 1.2\,T] \\ &= 1.5[E(S_1) + E(S_2)] - 1.2\,E(T) \\ &= 1.5(1.17+1.17) - 1.2(2.34) \\ &= 3.51 - 2.808 \\ &= 0.702 \end{aligned}$$

$$\begin{aligned} Var(Y) &= Var[1.5\,(S_1 + S_2) - 1.2\,T] \\ &= 1.5^2[Var(S_1) + Var(S_2)] + 1.2^2\,Var(T) \\ &= 1.5^2(0.003125 + 0.003125) + 1.2^2(0.00625) \\ &= 0.0140625 + 0.009 \\ &= 0.0230625 \end{aligned}$$

$Y \sim N\ (0.702,\ 0.0230625).$

Using G. C., $P(Y \geq 0.50) = 0.908264$

$\approx$ **0.908 (3 sig. fig)**



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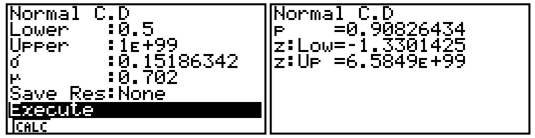

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