2012 AM P2. (v1.2)

1 $y = 2\sin x - 3\cos x$, $0 \leq x \leq \pi$.

(i) $\frac{dy}{dx} = 2\cos x + 3\sin x$.

$\frac{d^2y}{dx^2} = -2\sin x + 3\cos x$.

(ii) Stationary pts when $\frac{dy}{dx} = 0$.

$2\cos x + 3\sin x = 0$.

$\tan x = -\frac{2}{3}$.

basic $\measuredangle = \tan^{-1}(\frac{2}{3})$

$= 0.5880$

$\therefore x = 2.55$. (Cor to 3 sig fig)

(iii) when $x = 2.55$, $\frac{d^2y}{dx^2} = -2\sin(2.55) + 3\cos(2.55)$

$= -3.6055$

< 0.

$\therefore$ H is a maximum pt at $x = 2.55$.

2012 AM P2.

2 (i) $\cos 3x + \cos x = 2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}$ (using factor formula).
$= 2\cos 2x \cos x$

$\therefore A = 2$ and $B = 2$.

(ii) Given $\cos 3x + 2\cos x = 0$, for $0 \leq x \leq 180°$.
$\cos 3x + \cos x + \cos x = 0.$
$2\cos 2x \cos x + \cos x = 0$ {using result (i)}.
$\cos x\,[2\cos 2x + 1] = 0.$
$\cos x = 0$ or $\cos 2x = -\frac{1}{2}$.
$x = 90°$ basic $\measuredangle = \cos^{-1}(\frac{1}{2})$.
$= 60°$.
$2x = 120°, 240°$
$x = 60°, 120°$.

$\therefore x = 60°, 90°, 120°$ #.

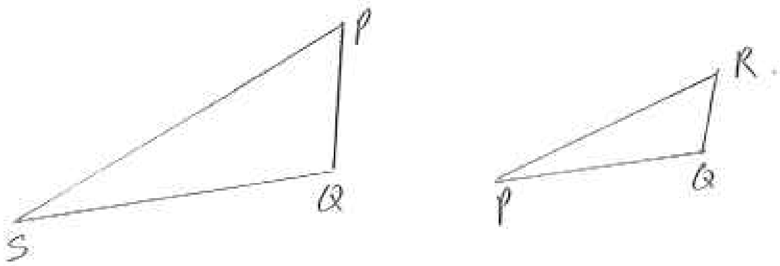
3 (i) $\frac{20}{3x^2+8x-3} = \frac{20}{(3x-1)(x+3)}$
Let $\frac{20}{(3x-1)(x+3)} = \frac{A}{3x-1} + \frac{B}{x+3}$, where A, B are constants.
$20 = A(x+3) + B(3x-1).$
when $x = -3$, $20 = 0 + B(-10).$
$B = -2.$
when $x = \frac{1}{3}$, $20 = A(\frac{10}{3}) + 0.$
$A = 6.$
$\therefore \frac{20}{(3x-1)(x+3)} = \frac{6}{3x-1} - \frac{2}{x+3}$ #

(ii) $\int \frac{20}{(3x-1)(x+3)}\,dx = \int\left(\frac{6}{3x-1} - \frac{2}{x+3}\right)dx$
$= 2\ln(3x-1) - 2\ln(x+3) + C.$

(iii) $\int_2^7 \frac{20}{(3x-1)(x+3)}\,dx = \left[2\ln(3x-1) - 2\ln(x+3)\right]_2^7.$
$= 2\ln 20 - 2\ln(10) - 2\ln 5 + 2\ln 5$
$= 2[\ln 20 - \ln 10].$
$= 2\ln\frac{20}{10}$
$= 2\ln 2.$
≈ 1.39 (corr to 3 sig fig)

2012 AM P2.

4



(i) In $\triangle PQS$ and $\triangle RQP$,

$\measuredangle PSQ = \measuredangle RPQ$ (alt segment thm).

$\measuredangle SPQ = \measuredangle PRQ$ (alt segment thm).

$\therefore \triangle PQS$ is similar to $\triangle RQP$. (Proved)

(ii) Since $\triangle PQS$ is similar to $\triangle RQP$

$$\frac{QS}{QP} = \frac{QP}{QR}$$

$\therefore (QS)(QR) = (QP)^2$ (Proved).

(iii). Let $\measuredangle SAP = \theta$. and $\measuredangle PBR = \alpha$.

In Quad APQS, $\measuredangle PQS = 180° - \theta$ (opp $\measuredangle$s in cyclic Quad)

In Quad PBRQ, $\measuredangle RQP = 180° - \alpha$.

Since $\triangle PQS$ is similar to $\triangle RQP$ (Proven in part(i)).

$\therefore \measuredangle PQS = \measuredangle RQP$

$180° - \theta = 180° - \alpha$

$\theta = \alpha$.

$\therefore \measuredangle SAP = \measuredangle PBR$. (Proved).

Alternative Method

$\measuredangle SAP = \measuredangle SPR$ (alt segment thm)

$\measuredangle PBR = \measuredangle SPR$ (alt segment thm)

$\therefore \measuredangle SAP = \measuredangle PBR$ (proved)

2012 AM P2.

5 Given $y = Ae^{2x} + Be^{-x}$. —— ①

Sub $x=0$, $y=4$.

$4 = A + B$. —— ②

From ①, $\frac{dy}{dx} = 2Ae^{2x} - Be^{-x}$

Given $\frac{dy}{dx} = -1$ when $x=0$.

$-1 = 2A - B$. —— ③.

② + ③ $\quad 3 = 3A$

$A = 1$ #.

Sub $A=1$ into ②,

$B = 3$ #.

(ii) Sub $A=1$ and $B=3$ into ①.

$y = e^{2x} + 3e^{-x}$.

$\int y\,dx = \int (e^{2x} + 3e^{-x})\,dx$

$= \frac{1}{2}e^{2x} - 3e^{-x} + C$ #.

$\int_0^1 y\,dx = \left[\frac{1}{2}e^{2x} - 3e^{-x}\right]_0^1$

$= \frac{1}{2}e^2 - 3e^{-1} - \frac{1}{2} + 3$

$= \frac{1}{2}e^2 - \frac{3}{e} + 2\frac{1}{2}$

$= 5.090$

≈ 5.1 (cor to 1 dec pl).

6 (a) (i).

$$\log_8 x^3 = \log_4 u.$$

$$\frac{\lg x^3}{\lg 8} = \frac{\lg u}{\lg 4}$$

$$\frac{\cancel{3}\lg x}{\cancel{3}\lg 2} = \frac{\lg u}{2\lg 2}.$$

$$\lg x = \tfrac{1}{2}\lg u.$$

$$2\lg x = \lg u$$

$$\lg u = \lg x^2$$

$$u = x^2 \ \#.$$

(a) (ii).

$$\log_4 (x^2+5x) - \log_8 x^3 = \frac{1}{\log_3 4}$$

$$\frac{\lg(x^2+5x)}{\lg 4} - \frac{3\lg x}{\lg 8} = \frac{\lg 3}{\lg 4}.$$

$$\frac{\lg(x^2+5x)}{2\lg 2} - \frac{\cancel{3}\lg x}{\cancel{3}\lg 2} = \frac{\lg 3}{2\lg 2}$$

$$\lg(x^2+5x) - 2\lg x = \lg 3.$$

$$\lg(x^2+5x) = \lg 3 + \lg x^2.$$

$$\lg(x^2+5x) = \lg 3x^2.$$

$$x^2+5x = 3x^2.$$

$$2x^2 - 5x = 0.$$

$$x(2x-5) = 0.$$

$$x = 0 \quad \text{or} \quad x = \frac{5}{2}.$$

Since $x > 0$, then $x = \frac{5}{2}$ #.

(b) Given

$$e^y(e^y - 2) = 15.$$

$$e^{2y} - 2e^y - 15 = 0.$$

$$(e^y - 5)(e^y + 3) = 0.$$

$e^y = 5$ or $e^y = -3$ (reject as $e^y > 0$).

$$\ln e^y = \ln 5$$

$$y = \ln 5.$$

$$\approx 1.61 \text{ (corr to 3 sig figs)}.$$

7. (i). Let $f(x) = 2x^4 + 3x^3 + a(x^2 + x) + b$.

Given $2x^2 + 3x - 2 = (2x - 1)(x + 2)$ is factor of $f(x)$

Then $f(\frac{1}{2}) = 0$ and $f(-2) = 0$.

For $f(\frac{1}{2}) = 0$,

$$2(\tfrac{1}{2})^4 + 3(\tfrac{1}{2})^3 + a((\tfrac{1}{2})^2 + \tfrac{1}{2}) + b = 0.$$

$$\tfrac{1}{8} + \tfrac{3}{8} + \tfrac{3}{4}a + b = 0.$$

$$\tfrac{1}{2} + \tfrac{3}{4}a + b = 0.$$

$$3a + 4b = -2. \quad \text{——} (1)$$

for $f(-2) = 0$.

$$2(-2)^4 + 3(-2)^3 + a((-2)^2 + (-2)) + b = 0.$$

$$32 - 24 + 2a + b = 0.$$

$$8 + 2a + b = 0.$$

$$b = -2a - 8. \quad \text{——} (2)$$

Sub (2) into (1).

$$3a + 4(-2a - 8) = -2.$$

$$3a - 8a - 32 = -2.$$

$$-5a = 30$$

$$a = -6 \#.$$

Put $a = -6$ into (2),

$$b = 4 \#$$

(ii) Let $f(x) = 2x^4 + 3x^3 - 6(x^2 + x) + 4 = (2x^2 + 3x - 2)(Ax^2 + Bx + C)$
$= (2x - 1)(x + 2)(Ax^2 + Bx + C)$.

$\therefore 2x^4 + 3x^3 - 6(x^2 + x) + 4 = (2x - 1)(x + 2)(Ax^2 + Bx + C)$.

Compare coeff of x^4: $A = 1$.

Compare constant term: $4 = -2C. \Rightarrow C = -2$.

When $x = 1$, $2 + 3 - 6(2) + 4 = (1)(3)[1 + B - 2]$.

$$\therefore B = 0.$$

2012 AM P2.

7 (ii) Continue part (ii).

$\therefore \quad 2x^4 + 3x^3 - 6(x^2+x) + 4 = 0.$

$(2x-1)(x+2)(x^2-2) = 0.$

$2x-1=0$ or $x+2=0$ $\quad x^2-2=0.$

$x = \frac{1}{2}$ $\quad x=-2.$ $\quad x = \pm\sqrt{2}.$

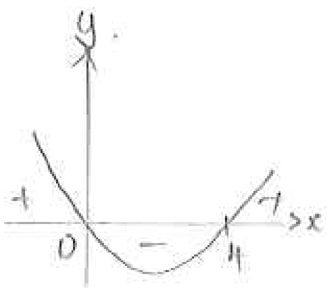
$\therefore \quad x = \frac{1}{2}, -2, \sqrt{2}, -\sqrt{2}.$

2012 AM P1

8(i). Given $y = 2x - x^2$ —— ①
$y = mx + 1$ —— ②

① = ②. $mx + 1 = 2x - x^2$.
$x^2 + mx - 2x + 1 = 0$.
$x^2 + (m-2)x + 1 = 0$.

Given they do not intersect, then
$b^2 - 4ac < 0$.
$(m-2)^2 - 4(1)(1) < 0$.
$(m-2)^2 - 4 < 0$.
$(m-2-2)(m-2+2) < 0$.
$(m-4)(m) < 0$.
$\therefore \quad 0 < m < 4$

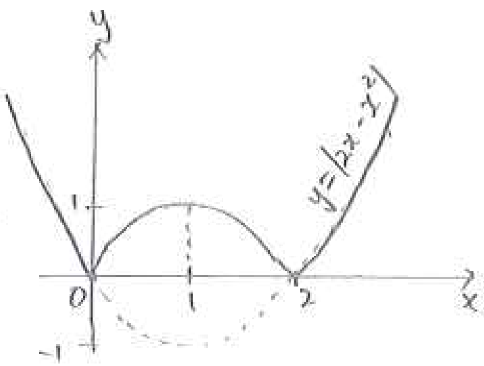


$\therefore$ The set of values of $m = \{m : 0 < m < 4\}$ #.

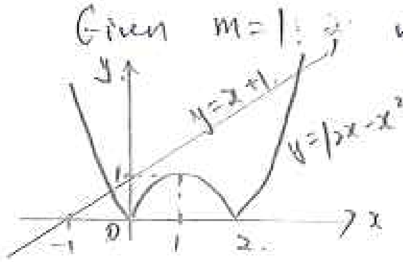
(ii) $y = |2x - x^2| = |x(x-2)|$

When $x = 1$, $y = |1(1-2)| = 1$.
Note $x = 1$ is the symmetrical line.

When $y = 0$, $x = 2$ or $x = 0$.

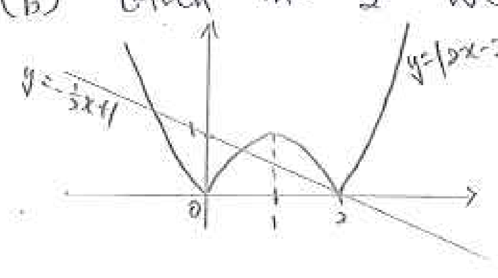


(iii). (a) Given $m = 1$, we have $|2x - x^2| = x + 1$.



$\therefore$ No. of solution $= 2$ #

(b) Given $m = -\frac{1}{2}$ we have $|2x - x^2| = -\frac{1}{2}x + 1$.



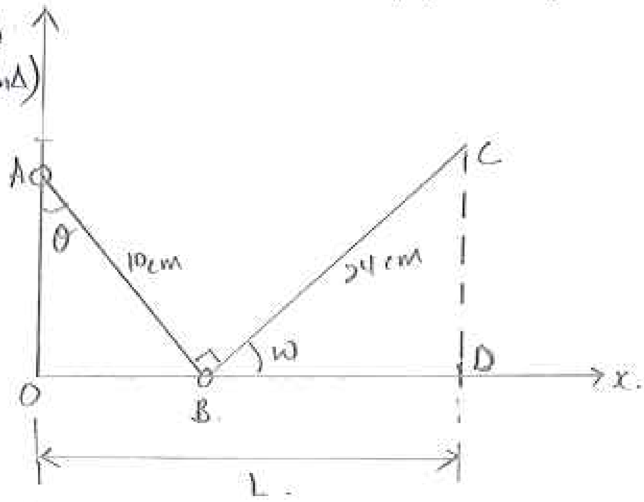
$\therefore$ No. of solutions $= 3$ #

2012 AM P2.

9 (i). In ΔOAB,

$\angle OBA = 180^\circ - 90^\circ - \theta$ - (Sum of ∠s in Δ)

$= 90^\circ - \theta$



∴ $\angle w + \angle ABC + \angle OBA = 180^\circ$

(adj ∠s on a str. line.

$\angle w = 180^\circ - 90^\circ - (90^\circ - \theta)$

$= \theta$.

∴ In ΔOBA,

$\sin\theta = \frac{OB}{AB}$.

$OB = 10\sin\theta$.

In ΔBCD,

$\cos\theta = \frac{BD}{BC}$.

$BD = 24\cos\theta$.

∴ $L = OB + BD$

$= 10\sin\theta + 24\cos\theta$ (shown)#.

(ii). $L = 10\sin\theta + 24\cos\theta = R\cos(\theta - \alpha)$.

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$.

$= R\cos\alpha\cos\theta + R\sin\alpha\sin\theta$.

∴ $R\sin\alpha = 10$. —— ①

$R\cos\alpha = 24$ —— ②.

①^2+②^2 $R^2[\sin^2\alpha + \cos^2\alpha] = 10^2 + 24^2$.

$R = 26$ (Since $R > 0$).

①/②. $\frac{R\sin\alpha}{R\cos\alpha} = \frac{10}{24}$

$\tan\alpha = \frac{5}{12}$.

$\alpha = 22.62^\circ$.

∴ $L = 10\sin\theta + 24\cos\theta = 26\cos(\theta - 22.6^\circ)$.#.

9 (iii) Greatest possible value of $L = 26$.
when $\cos(\theta - 22.62^\circ) = 1$

$0 < \theta < 90^\circ$
$-22.62^\circ < \theta - 22.62^\circ < 67.38^\circ$

$\therefore \theta - 22.62^\circ = 0$.
$\theta = 22.62^\circ$
$\approx 22.6^\circ$ (corr to 1 dec pl).

(iv). Given $L = 20$, then.
$26 \cos(\theta - 22.62^\circ) = 20$.
$\cos(\theta - 22.62^\circ) = \frac{20}{26}$.
basic $\measuredangle = \cos^{-1}\left(\frac{20}{26}\right)$.
$= 39.72^\circ$.

$\therefore \theta - 22.62^\circ = 39.72^\circ$.
$\theta = 62.34^\circ$
$\approx 62.3^\circ$ (corr to 1 dec pl)

2012 AM P2.

10 : $A(8, k)$, $B(3, 6)$, $C(1, 2)$.

Given M is midpoint of AB.

$$M = \left(\frac{8+3}{2}, \frac{k+6}{2}\right)$$
$$= \left(\frac{11}{2}, \frac{k+6}{2}\right).$$

Since M lies on $y + 3x = 25$.

when $x = \frac{11}{2}$, and $y = \frac{k+6}{2}$,

$$\frac{k+6}{2} + 3\left(\frac{11}{2}\right) = 25.$$
$$\frac{k+6}{2} = \frac{17}{2}.$$
$$k + 6 = 17$$
$$k = 11 \text{ (shown)}.$$

(ii) Gradient of BC $= \frac{6-2}{3-1} = \frac{4}{2} = 2.$ = Gradient of AD.

Equation of line AD :

$$(y - 11) = 2(x - 8).$$
$$y = 2x - 5 \text{ —— (1)}.$$

Line MN : $y + 3x = 25$ —— (2).

Sub (1) into (2)

$$2x - 5 + 3x = 25.$$
$$5x = 30$$
$$x = 6.$$

Put $x = 6$ into (1), $y = 7$.

$\therefore N(6, 7)$ #

Gradient of CD $= -\frac{1}{\text{Gradient of AD}} = -\frac{1}{2}.$

Equation of line CD :

$$y - 2 = -\frac{1}{2}(x - 1).$$
$$2y - 4 = -x + 1.$$
$$2y = -x + 5. \text{ —— (3)}.$$

2012 AM P2.

10 (ii) Sub ① into ③.

$$2(2x-5) = -x+5$$
$$4x - 10 = -x+5$$
$$5x = 15$$
$$x = 3$$

Put $x = 3$ into ①.

$$y = 2(3) - 5 = 1$$

$\therefore D(3, 1)$.

(iii) $A(8, 11)$, $M\left(\frac{11}{2}, \frac{17}{2}\right)$, $N(6, 7)$.

$$\text{Area of } \triangle AMN = \frac{1}{2}\begin{vmatrix} 8 & \frac{11}{2} & 6 & 8 \\ 11 & \frac{17}{2} & 7 & 4 \end{vmatrix}$$
$$= \frac{1}{2}\left[8\times\frac{17}{2} + \frac{11}{2}\times 7 + 6\times 11 - 11\times\frac{11}{2} - 6\times\frac{17}{2} - 7\times 8\right]$$
$$= 2.5 \text{ units}^2$$

2012 AM P2.

11 (i) Given $y = \sqrt{5+4x} = (5+4x)^{1/2}$.

$$\frac{dy}{dx} = \frac{1}{2}(5+4x)^{-\frac{1}{2}}(4).$$
$$= 2(5+4x)^{-\frac{1}{2}}$$

When $x = 1$, gradient of tangent, $m_T = \frac{2}{\sqrt{9}} = \frac{2}{3}$.
gradient of normal, $m_N = -1/(\frac{2}{3}) = -\frac{3}{2}$.

$$y = [5+4(1)]^{1/2} = 3.$$

$B(1, 3)$.

$\therefore$ Eqn of BC :

$$y - 3 = -\frac{3}{2}(x-1).$$
$$2y - 6 = -3x + 3.$$
$$2y = -3x + 9.$$
$$y = -\frac{3}{2}x + \frac{9}{2}.$$

when $y=0$, $0 = -3x + 9$
$$3x = 9$$
$$x = 3.$$

$\therefore$ $C(3, 0)$ #.

(ii) To find A :
when $y = 0$, $0 = \sqrt{5+4x}$
$$5+4x = 0$$
$$x = -\frac{5}{4}.$$

Area of shaded region
$$= \int_{-5/4}^{1} \sqrt{5+4x}\, dx + \frac{1}{2}(3)(2).$$

$$= \left[\frac{1}{4} \times \frac{2}{3}(5+4x)^{3/2}\right]_{-5/4}^{1} + 3.$$

$$= \frac{1}{6}\left[9^{3/2} - 0\right] + 3.$$
$$= 7.5 \text{ units}^2.$$