V1.1 Amaths P2

1. $y = 2x^2 + kx + 6 - k$

(i) Given the curve lies completely above x-axis,

$b^2 - 4ac < 0$

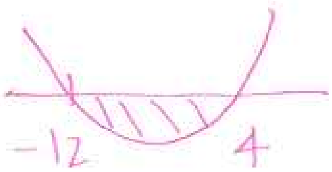
we have $a = 2$, $b = k$, $c = 6 - k$

$\Rightarrow (k)^2 - 4(2)(6-k) < 0$

$k^2 - 48 + 8k < 0$

$k^2 + 8k - 48 < 0$

$(k-4)(k+12) < 0$



Range of values of k = $-12 < k < 4$ #

(ii) Given $k = 2$, $y = 2x^2 + 2x + 4$ —— ①

$y = mx - 4$ —— ②

① = ②, $2x^2 + 2x + 4 = mx - 4$

$2x^2 + 2x - mx + 8 = 0$

Since the line is a tangent to the curve,

$b^2 - 4ac = 0$

we have $a = 2$, $b = 2 - m$, $c = 8$

$\Rightarrow (2-m)^2 - 4(2)(8) = 0$

$(2-m)^2 = 64$

$2 - m = 8$ or $2 - m = -8$

$m = -6$ $m = 10$ #

2. Given $f(x) = a\tan bx$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(i) $f(x) = 0$ when $x = \frac{\pi}{2}$,

$$a\tan\frac{\pi}{2}b = 0$$

$$\tan\frac{\pi}{2}b = 0$$

$$\frac{\pi}{2}b = 0, \pi, 2\pi$$

$$b = 0, 2, 4$$

$\therefore$ Smallest possible value of $b = 2$ #

(ii) let $y = a\tan 2x$

$$\frac{dy}{dx} = 2a\sec^2 2x$$

Given $\frac{dy}{dx} = 12$ when $x = \frac{\pi}{8}$,

$$12 = 2a\sec^2\frac{\pi}{4}$$

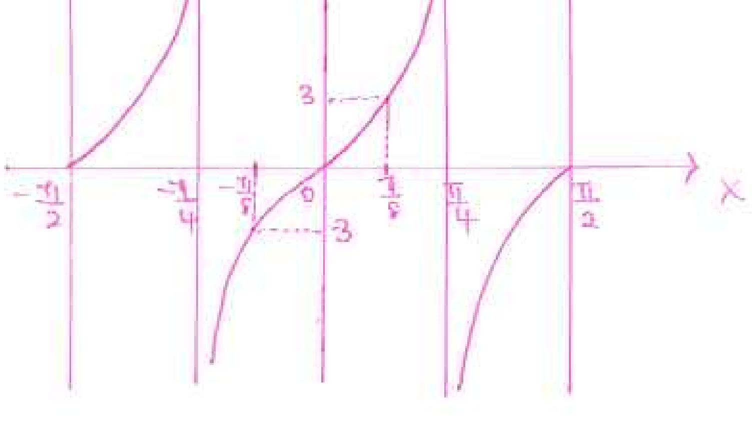
$$12 = \frac{2a}{\left(\cos\frac{\pi}{4}\right)^2}$$

$$2a = 12\left(\frac{1}{\sqrt{2}}\right)^2$$

$$2a = 6$$

$$a = 3 \text{ #}$$

(iii) $y = 3\tan 2x$



3. (i) let $f(x) = 2x^3 + ax + b$

$f(-2) = 0$

$\Rightarrow \quad 2(-2)^3 - 2a + b = 0$

$-16 - 2a + b = 0$

$b = 2a + 16 \quad \longrightarrow (1)$

$f(3) = 35$

$\Rightarrow \quad 2(3)^3 + 3a + b = 35$

$3a + b = -19 \quad \longrightarrow (2)$

Sub (1) into (2),

$3a + 2a + 16 = -19$

$5a = -35$

$a = -7$

Sub $a = -7$ into (1),

$b = 2(-7) + 16$

$= 2$

$\therefore \ a = -7$ and $b = 2$ #

3 (ii) $2x^3 - 7x + 2 = 0$

Since $(x+2)$ is a factor,

$\Rightarrow \quad 2x^3 - 7x + 2 = (x+2)(Ax^2 + Bx + C)$

comparing coefficient of x^3: $A = 2$

comparing coefficient of x^2: $0 = 2A + B$

$B = -4$

Comparing coefficient of x^0: $2 = 2C$

$C = 1$

$\Rightarrow \quad 2x^3 - 7x + 2 = (x+2)(2x^2 - 4x + 1) = 0$

$x + 2 = 0$ or $2x^2 - 4x + 1 = 0$

$\therefore \quad x = -2$

$$x = \frac{4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{8}}{4}$$

$$= \frac{4 \pm 2\sqrt{2}}{4}$$

$$= \frac{2 \pm \sqrt{2}}{2}$$

#

4. For $2x^2+4x+5=0$ with roots α and β,

$\Rightarrow \quad \alpha+\beta=-2$

$\alpha\beta=\frac{5}{2}$

(i) $\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

$=\frac{(-2)^2-2\left(\frac{5}{2}\right)}{\frac{5}{2}}$

$=-\frac{2}{5}$ (shown) #

(ii) For $x^2+ax+b=0$ with roots $\frac{\alpha}{\beta}+2$ and $\frac{\beta}{\alpha}+2$,

Sum of roots: $\frac{\alpha}{\beta}+2+\frac{\beta}{\alpha}+2=-a$

$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=-a-4$

$\frac{\alpha^2+\beta^2}{\alpha\beta}=-a-4$

Using the result in (i),

$-\frac{2}{5}=-a-4$

$a=-3\frac{3}{5}$ #

Product of roots: $\left(\frac{\alpha}{\beta}+2\right)\left(\frac{\beta}{\alpha}+2\right)=6$

$$1+\frac{2\alpha}{\beta}+\frac{2\beta}{\alpha}+4=6$$

$$\frac{2\alpha^2+2\beta^2}{\alpha\beta}=6-5$$

$$2\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)=6-5$$

Using the result in (i),

$$2\left(-\frac{2}{5}\right)=6-5$$

$$b=-\frac{4}{5}+5$$

$$=4\frac{1}{5}$$

$\therefore\ a=-3\frac{3}{5}$ and $b=4\frac{1}{5}$ #

5(a) $\log_3 x = 4+\log_{27} x$

$$\log_3 x = 4+\frac{\log_3 x}{\log_3 27}$$

$$\log_3 x = 4+\frac{\log_3 x}{3}$$

$$3\log_3 x = 12+\log_3 x$$

$$2\log_3 x = 12$$

$$\log_3 x = 6$$

$$\therefore x = 3^6$$

$$=729$$

#

5(b) $y = ax^n$ —— (1)

Sub (2, 40) into (1), $40 = a(2)^n$ —— (2)

Sub (3, 135) into (1), $135 = a(3)^n$ —— (3)

$\frac{(3)}{(2)}$, $\frac{135}{40} = \frac{a(3)^n}{a(2)^n}$

$\frac{135}{40} = \left(\frac{3}{2}\right)^n$

$\lg \frac{135}{40} = \lg \left(\frac{3}{2}\right)^n$

$\lg \frac{135}{40} = n \lg \left(\frac{3}{2}\right)$

$n = 3$ #

Sub $n=3$ into (2), $40 = a(2)^3$

$a = 5$ #

Sub $(4, k)$, $a=5$ and $n=3$ into (1),

$k = 5(4)^3$

$= 320$ #

$\therefore n=3$, $a=5$ and $k=320$ #

6. (i) Area of the lawn $= 400 \text{ m}^2$

$$2r(l) - \frac{1}{2}\pi(r)^2 = 400$$

$$2rl = 400 + \frac{1}{2}\pi r^2$$

$$2rl = \frac{800 + \pi r^2}{2}$$

$$l = \frac{800 + \pi r^2}{4r}$$ #

(ii) $P = l + l + 2r + \pi r$

$$= 2\left(\frac{800 + \pi r^2}{4r}\right) + 2r + \pi r$$

$$= \frac{800 + \pi r^2}{2r} + 2r + \pi r$$

$$= \frac{400}{r} + \frac{\pi r}{2} + 2r + \pi r$$

$$= \frac{3\pi r}{2} + 2r + \frac{400}{r}$$

$$= \left(\frac{3\pi}{2} + 2\right)r + \frac{400}{r}$$ (shown) #

(iii) $\frac{dP}{dr} = \frac{3\pi}{2} + 2 - \frac{400}{r^2}$

$$\frac{dP}{dr} = 0$$

$$\frac{400}{r^2} = \frac{3\pi}{2} + 2$$

$$r^2 = (400) \div \left(\frac{3\pi}{2} + 2\right)$$

$r = 7.7195$ or -7.7195 (rej)

≈ 7.72 m

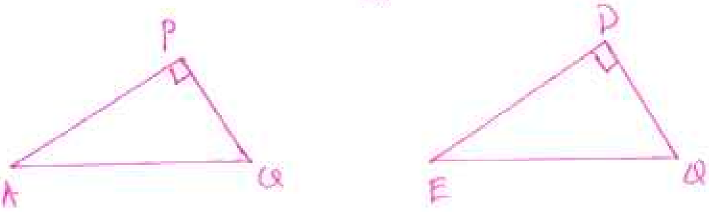
$\frac{dP}{dt} = 0$ at $r = 7.72$ and so P has a stationary value #

$$\frac{d^2p}{dr^2} = \frac{800}{r^3}$$

Sub $r = 7.7195$, $\frac{d^2p}{dr^2} = \frac{800}{(7.7195)^3} > 0$

$\therefore$ p is minimum. #

7. (i)



Consider $\triangle QAD$ and $\triangle QED$,

$QA = QE$ = radius

QD is common.

$\therefore$ $\triangle QAD$ is congruent to $\triangle QED$. (RHS)

Hence $AD = ED$ (shown).

(ii) $PA \times PE = (PD - AD)(PD + DE)$

$= (PD - AD)(PD + AD)$ $\quad (\because DE = AD)$

$= PD^2 - AD^2$ (shown) #

(iii) By tangent-secant theorem for C_2,

$PA \times PE = PR^2$

Since $PB = PR$ = radius of circle C_1.

$\therefore$ $PA \times PE = PB^2$

Using result from (ii), $PA \times PE = PB^2$

$\Rightarrow PD^2 - AD^2 = PB^2$ (shown) #

8 (i) $\dfrac{8x^2+3x+1}{x(2x+1)^2} = \dfrac{A}{x} + \dfrac{B}{2x+1} + \dfrac{C}{(2x+1)^2}$

$8x^2+3x+1 = A(2x+1)^2 + B(x)(2x+1) + C(x)$

Sub $x = -\frac{1}{2}$, $\quad \frac{3}{2} = -\frac{1}{2}C$

$C = -3$

Sub $x = 0$, $\quad 1 = A$

Sub $x = 1$, $\quad 12 = 1(9) + B(1)(3) - 3(1)$

$3B = 6$

$B = 2$

$\therefore \dfrac{8x^2+3x+1}{x(2x+1)^2} = \dfrac{1}{x} + \dfrac{2}{2x+1} - \dfrac{3}{(2x+1)^2}$ #

(ii) $\displaystyle\int \frac{8x^2+3x+1}{x(2x+1)^2}\,dx = \int\left[\frac{1}{x} + \frac{2}{2x+1} - \frac{3}{(2x+1)^2}\right]dx$

$\displaystyle = \int\left[\frac{1}{x} + \frac{2}{2x+1} - 3(2x+1)^{-2}\right]dx$

$\displaystyle = \ln x + \frac{2\ln(2x+1)}{2} - \frac{3(2x+1)^{-1}}{-2} + C$

$\displaystyle = \ln x + \ln(2x+1) + \frac{3}{2(2x+1)} + C$

#

9. (i) $y = 3\sin\frac{x}{2}$

$\frac{dy}{dx} = 3\left(\cos\frac{x}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{3}{2}\cos\frac{x}{2}$

Sub $x = \frac{4\pi}{3}$, $y = 3\sin\frac{2\pi}{3}$

$= 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$

$\therefore P\left(\frac{4\pi}{3}, \frac{3\sqrt{3}}{2}\right)$

and $\frac{dy}{dx} = \frac{3}{2}\cos\frac{2\pi}{3}$

$= -\frac{3}{4}$

$\Rightarrow$ Equation of PR : $y - \frac{3\sqrt{3}}{2} = -\frac{3}{4}\left(x - \frac{4\pi}{3}\right)$

$y = -\frac{3}{4}x + \pi + \frac{3\sqrt{3}}{2}$

When PR meets the x-axis at R, sub $y=0$,

$\Rightarrow$ $-\frac{3}{4}x + \pi + \frac{3\sqrt{3}}{2} = 0$

$\frac{3}{4}x = \pi + \frac{3\sqrt{3}}{2}$

$x = \frac{4}{3}\pi + 2\sqrt{3}$

$\therefore R\left(\frac{4}{3}\pi + 2\sqrt{3}, 0\right)$

$\therefore$ length of QR $= \frac{4}{3}\pi + 2\sqrt{3} - \frac{4}{3}\pi$

$= 2\sqrt{3}$. #

9 (v) Area of shaded region

$= \int_0^{\frac{4\pi}{3}} 3\sin\frac{x}{2}\,dx + \frac{1}{2}(PQ)(QR)$

$= \left[-6\cos\frac{x}{2}\right]_0^{\frac{4\pi}{3}} + \frac{1}{2}\left(\frac{3\sqrt{3}}{2}\right)(2\sqrt{3})$

$= \left[-6\cos\frac{2\pi}{3} + 6\cos 0\right] + 4.5$

$= 3 + 6 + 4.5$

$= 13.5 \text{ units}^2$ ✻

10. (i) $\cos 15° - \cos 75°$ * Using factor formula

$= -2\sin\left(\frac{15° + 75°}{2}\right)\sin\left(\frac{15° - 75°}{2}\right)$

$= -2\sin 45° \sin(-30°)$

$= 2\sin 45° \sin 30°$

$= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{2}}{2}$ ✻

OR. $\cos 15° - \cos 75°$ * Using Additional Formulla.

$= \cos(45° - 30°) - \cos(45° + 30°)$

$= \cos 45°\cos 30° + \sin 45°\sin 30° - \cos 45°\cos 30° + \sin 45°\sin 30°$

$= 2\sin 45°\sin 30°$

$= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{2}}{2}$ ✻

10 (ii) $\cos 15^\circ + \cos 75^\circ$ * Using factor formula

$= 2\cos\left(\frac{15^\circ + 75^\circ}{2}\right)\cos\left(\frac{15^\circ - 75^\circ}{2}\right)$

$= 2\cos 45^\circ \cos(-30^\circ)$

$= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{6}}{2}$ #

OR

$\cos 15^\circ + \cos 75^\circ$ * Using Addition formula.

$= \cos(45^\circ - 30^\circ) + \cos(45^\circ + 30^\circ)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ + \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$= 2\cos 45^\circ \cos 30^\circ$

$= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{6}}{2}$ #

(iii) LHS $= \cos^2 15^\circ - \cos^2 75^\circ$

$= (\cos 15^\circ)^2 - (\cos 75^\circ)^2$

$= (\cos 15^\circ + \cos 75^\circ)(\cos 15^\circ - \cos 75^\circ)$

Using the results in (i) and (ii),

$= \frac{\sqrt{6}}{2}\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{12}}{4}$

$= \frac{2\sqrt{3}}{4}$

$= \frac{\sqrt{3}}{2} =$ RHS (shown) #

10 (iv) $\sin\theta = \cos 75^\circ$

$\sin\theta = \sin 15^\circ$

$\theta = 15^\circ$.

(v) Using result from (iv),

$\sin 15^\circ = \cos 75^\circ$

$\sin^2 15^\circ = \cos^2 75^\circ$ — ①

Using result from (iii),

$\cos^2 15^\circ - \cos^2 75^\circ = \frac{\sqrt{3}}{2}$ — ②

Sub ① into ②,

$\cos^2 15^\circ - \sin^2 15^\circ = \frac{\sqrt{3}}{2}$

$\cos^2 15^\circ - (1 - \cos^2 15^\circ) = \frac{\sqrt{3}}{2}$

$\cos^2 15^\circ - 1 + \cos^2 15^\circ = \frac{\sqrt{3}}{2}$

$2\cos^2 15^\circ = \frac{\sqrt{3}}{2} + 1$

$\cos^2 15^\circ = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right)$

$= \frac{\sqrt{3}}{4} + \frac{1}{2}$

#

11. (i) Centre C = midpt of AB

$= \left(\frac{1+7}{2}, \frac{1+9}{2} \right)$

$= (4, 5)$ ※

radius $= \sqrt{(4-1)^2 + (5-1)^2}$

$= 5$ units ※

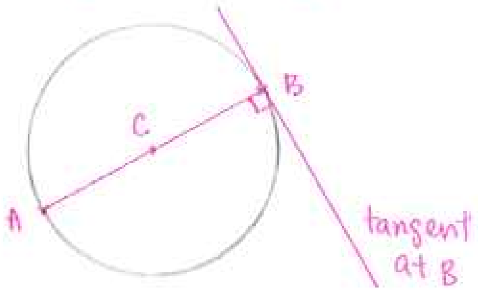
(ii) Equation of circle : $(x-4)^2 + (y-5)^2 = 5^2$

$(x-4)^2 + (y-5)^2 = 25$

$\Rightarrow x^2 - 8x + y^2 - 10y + 16 = 0$ ※

(iii) Gradient of AB $= \frac{9-1}{7-1}$

$= \frac{4}{3}$



Gradient of tangent at B $= -\frac{3}{4}$

Equation of tangent at B: $y - 9 = -\frac{3}{4}(x - 7)$

$y = -\frac{3}{4}x + 14\frac{1}{4}$

$4y = -3x + 57$

$4y + 3x = 57$ (shown) ※

(iv) Given D is the lowest point on circle

$\Rightarrow$ CD is vertical and perpendicular to x-axis.

Since y-coordinate of C is 5 and radius is 5 units,

$\Rightarrow$ D lies on x-axis. ※

11 (v) tangent to the circle at B

$\Rightarrow 4y + 3x = 57$ — (1)

tangent to circle at D $\Rightarrow y = 0$ — (2)

sub (2) into (1),

$3x = 57$

$x = 19$

$\therefore$ Coordinates of the point which the tangents to the circle at B and D intersect

$= (19, 0)$ ※