

ADDITIONAL MATHEMATICS
Paper 2 Suggested Solutions

4038/02
October/November 2009

1. **Topic: Further Trigonometric Identities**

(i) $\sin(A - B) = \frac{3}{8}$

$$\sin A \cos B - \cos A \sin B = \frac{3}{8}$$

$$\frac{5}{8} - \cos A \sin B = \frac{3}{8}$$

$$\cos A \sin B = \frac{5}{8} - \frac{3}{8}$$

$$= \frac{1}{4}$$

Addition Formula:
 $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \frac{5}{8} + \frac{1}{4}$$

$$= \frac{7}{8}$$

Addition Formula:
 $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(iii) $\frac{\tan A}{\tan B} = \frac{\sin A}{\cos A} \div \frac{\sin B}{\cos B}$

$$= \frac{\sin A}{\cos A} \times \frac{\cos B}{\sin B}$$

$$= \frac{\sin A \cos B}{\cos A \sin B}$$

$$= \frac{\frac{5}{8}}{\frac{1}{4}}$$

Using ans from (ii)

$$= 2\frac{1}{2}$$

Using ans from (i)

2. **Topic: Partial Fractions and Integration**

(i) $\frac{7}{2x^2-x-6} = \frac{7}{(2x+3)(x-2)}$
 $\Rightarrow \frac{7}{(2x+3)(x-2)} = \frac{A}{2x+3} + \frac{B}{x-2}$

Distinct linear factor:
 $\frac{P(x)}{(ax+b)(cx-d)} = \frac{A}{ax+b} + \frac{B}{cx-d}$

$$7 = A(x-2) + B(2x+3)$$

When $x = 2$, $7 = 0 + B(7)$

$$B = 1$$

When $x = -\frac{3}{2}$, $7 = A\left(-\frac{3}{2} - 2\right) + 0$

$$A = -2$$

$$\therefore \frac{7}{2x^2-x-6} = -\frac{2}{2x+3} + \frac{1}{x-2}$$

***Hence question:**
From part (i)

(ii) $\int_3^9 \frac{7}{2x^2-x-6} dx = \int_3^9 \left(-\frac{2}{2x+3} + \frac{1}{x-2}\right) dx$

$$= \left[-\frac{2\ln(2x+3)}{2} + \ln(x-2)\right]_3^9$$

$$= [-\ln(21) + \ln 7] - [-\ln 9 + \ln 1]$$

$$\approx -1.0986 - (-2.1972)$$

$$= 1.0986$$

$$\approx \mathbf{1.10 \text{ (3 sig. fig.)}}$$

3. Topic: Indices, Logarithms and Factor Theorem

(i) $u = 2^x$

$$8^x - 2^{x+2} = 15$$

$$(2^3)^x - 2^x \times 2^2 = 15$$

$$(2^x)^3 - 4 \times 2^x = 15$$

$$\therefore u^3 - 4u - 15 = 0$$

(ii) Let $f(u) = u^3 - 4u - 15$

When $u = 3$, $f(3) = 3^3 - 4(3) - 15$

$$= 0$$

\therefore By factor theorem, $(u - 3)$ is a factor.

$$f(u) = (u - 3)(u^2 + bu + 5)$$

Compare coefficients of u : $-4 = 5 - 3b$

$$3b = 9$$

$$b = 3$$

$$\therefore f(u) = (u - 3)(u^2 + 3u + 5) = 0$$

$$\Rightarrow u - 3 = 0 \quad \text{or} \quad u^2 + 3u + 5 = 0$$

$$u = 3$$

$$u = \frac{-3 \pm \sqrt{3^2 - 4(1)(5)}}{2}$$

$$= \frac{-3 \pm \sqrt{-11}}{2} \text{ (rejected)}$$

$\therefore u = 3$ is the only real solution of this equation (Shown).

(iii) $u = 3$

$$\Rightarrow 2^x = 3$$

$$\lg 2^x = \lg 3$$

$$x \lg 2 = \lg 3$$

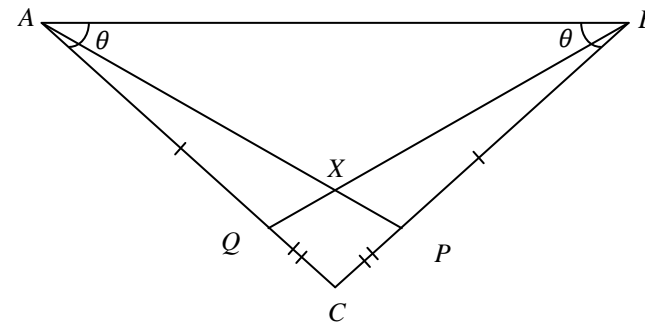
$$\log_a (x)^n = n \log_a x$$

$$x = \frac{\lg 3}{\lg 2}$$

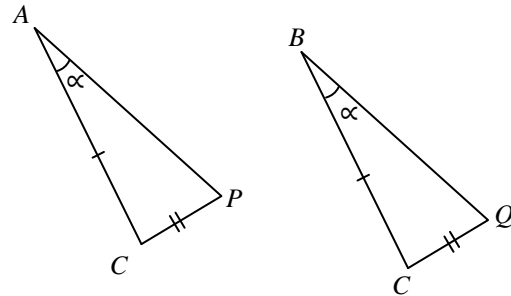
$$\approx 1.584$$

$$\approx 1.58 \text{ (3 sig. fig.)}$$

4. Topic: Plane Geometry



- (i) Given $\triangle ABC$ is an isosceles \triangle ,
 Let $\angle CAB = \angle CBA = \theta$



In $\triangle ACP$ and $\triangle BCQ$, $AC = BC$ (Given)
 $CP = CQ$ (Given)
 $\angle ACP = \angle BCQ$ (Common)

$\Rightarrow \triangle ACP$ is congruent to $\triangle BCQ$ (SAS).

Hence $\angle CAP = \angle CBQ = \alpha$

$\Rightarrow \angle XAB = \angle CAB - \angle CAP = \theta - \alpha$

and $\angle XBA = \angle CBA - \angle CBQ = \theta - \alpha$

$\therefore \angle XAB = \angle XBA$

Hence $\triangle XAB$ is isosceles \triangle . (Shown)

- (ii) Since $\triangle XAB$ is isosceles \triangle , $XA = XB = a$ (1)

Since $\triangle ACP$ is congruent to $\triangle BCQ$ (using result from part (i)),

$$AP = BQ$$

$$AX + XP = BX + XQ$$

$$a + XP = a + XQ \text{ (using result from part (i))}$$

$$XP = XQ$$

$$PX = QX \text{ (Shown)}$$

5. Topic: Binomial Expansion

$$\begin{aligned} \text{(i)} \quad \left(2 - \frac{x}{4}\right)^n &= 2^n + {}^n C_1 (2)^{n-1} \left(-\frac{x}{4}\right)^1 + {}^n C_2 (2)^{n-2} \left(-\frac{x}{4}\right)^2 + \dots \\ &= 2^n + n(2)^{n-1} \left(-\frac{x}{2^2}\right) + \frac{n(n-1)}{2} (2)^{n-2} \left(\frac{x^2}{2^4}\right) + \dots \\ &= 2^n - n2^{n-3}x + n(n-1)(2)^{n-7}x^2 + \dots \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad (1+x)\left(2 - \frac{x}{4}\right)^n &= a + bx^2 \\ \Rightarrow (1+x)[2^n - n2^{n-3}x + n(n-1)(2)^{n-7}x^2 + \dots] &= a + bx^2 \\ 2^n - n2^{n-3}x + n(n-1)(2)^{n-7}x^2 + 2^n x - n2^{n-3}x^2 &= a + bx^2 \end{aligned}$$

Compare coefficients of x^0 : $2^n = a$ (1)

Compare coefficients of x^1 : $-n2^{n-3} + 2^n = 0$

$$2^{n-3}[-n + 2^3] = 0$$

$$2^{n-3} = 0 \text{ (reject)} \quad \text{or} \quad -n + 8 = 0$$

$$n = 8$$

(iii) Compare coefficients of x^2 : $n(n-1)(2)^{n-7} - n2^{n-3} = b$ (2)

Sub $n = 8$ into (1), $a = 2^8$

$$= 256$$

Sub $n = 8$ into (2), $b = 8(7)2^1 - 8(2)^5$

$$= -144$$

Expanding $(a+b)^n$:
 $T_{r+1} = \binom{n}{r} a^{n-r} b^r$
 Given in formula sheet:
 $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
 $= \frac{n(n-1)\dots(n-r+1)}{r!}$
 $\therefore {}^n C_1 = \binom{n}{1} = n$
 ${}^n C_2 = \binom{n}{2} = \frac{n(n-1)}{2!}$

6. Topic: Trigonometric Functions and Area under curve

(i) $y = 1 + 2\cos x$

When $y = 0$, $1 + 2\cos x = 0$

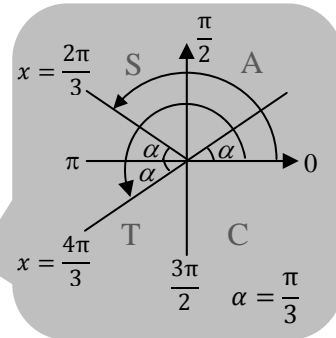
$$\cos x = -\frac{1}{2}$$

$$\text{Basic } \angle \alpha = \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3}$$

$$\therefore x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}, \frac{4\pi}{3}$$



\therefore x -coordinate of A is $\frac{2\pi}{3}$ (Shown) and x -coordinate of B is $\frac{4\pi}{3}$.

(ii) Area of the shaded region

$$= \int_0^{\frac{2\pi}{3}} (1 + 2\cos x) dx - \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2\cos x) dx$$

$$= [x + 2\sin x]_0^{\frac{2\pi}{3}} - [x + 2\sin x]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= \left[\frac{2\pi}{3} + 2\sin\frac{2\pi}{3} - 0\right] - \left[\frac{4\pi}{3} + 2\sin\frac{4\pi}{3} - \frac{2\pi}{3} - 2\sin\frac{2\pi}{3}\right]$$

$$= \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) - \frac{4\pi}{3} - 2\left(-\frac{\sqrt{3}}{2}\right) + \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3} + \sqrt{3} + \sqrt{3}$$

$$\approx 5.196$$

$$\approx \mathbf{5.20 \text{ units}^2} \text{ (3sig. fig.)}$$

Using values of A and B found in (i) as the limits

7. Topic: Modulus Functions

(i) $y = |3x - 5| - 2$

When $x = 0$, $y = |0 - 5| - 2$
 $= 3$

When $y = 0$, $|3x - 5| - 2 = 0$

$$3x - 5 = 2 \quad \text{or} \quad 3x - 5 = -2$$

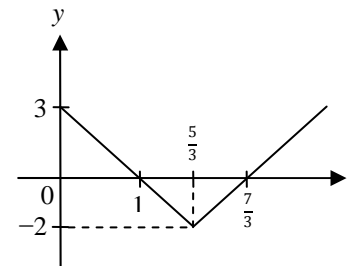
$$3x = 7 \quad \quad \quad 3x = 3$$

$$x = \frac{7}{3} \quad \quad \quad x = 1$$

$$= 2\frac{1}{3}$$

\therefore Coordinates of all the points meeting the axes are $(0, 3)$, $(2\frac{1}{3}, 0)$ and $(1, 0)$.

(ii) $y = |3x - 5| - 2$



(iii) $x = |3x - 5| - 2$

$$x + 2 = |3x - 5|$$

$$x + 2 = 3x - 5 \quad \text{or} \quad -(x + 2) = 3x - 5$$

$$2x = 7 \quad \quad \quad -x - 2 = 3x - 5$$

$$x = \mathbf{3.5} \quad \quad \quad 4x = 3$$

Solving modular equations:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

$$x = \mathbf{0.75}$$

8. **Topic: Applications of Differentiation (Kinematics)**

(i) $s = 400\left(1 - e^{-\frac{t}{10}}\right) - 16t$

$$v = \frac{ds}{dt}$$

$$= 400\left(\frac{1}{10}e^{-\frac{t}{10}}\right) - 16$$

$$= 40e^{-\frac{t}{10}} - 16$$

(ii) $a = \frac{dv}{dt}$

$$= 40\left(-\frac{1}{10}\right)e^{-\frac{t}{10}}$$

$$= -4e^{-\frac{t}{10}}$$

"t seconds after passing A"

(iii) At A, $t = 0$, $\therefore v = 40e^0 - 16$

$$= 24 \text{ m/s}$$

"coming to a rest at a point B"

(iv) At B, $v = 0$, $40e^{-\frac{t}{10}} - 16 = 0$

$$40e^{-\frac{t}{10}} = 16$$

$$e^{-\frac{t}{10}} = 0.4$$

$$\ln e^{-\frac{t}{10}} = \ln 0.4$$

$$-\frac{t}{10} = \ln 0.4$$

$$t = 9.1629$$

$$\approx 9.163 \text{ seconds (Shown)}$$

(v) Total distance = $400\left(1 - e^{-\frac{9.163}{10}}\right) - 16(9.163)$

$$= 93.393 \text{ m}$$

Sub $t = 9.163$ from part (iv) into s

Average speed of the motorcycle for the journey from A to B

$$= \frac{\text{Total distance}}{\text{Total time taken}}$$

$$= \frac{93.393}{9.163}$$

$$\approx 10.192$$

$$\approx 10.2 \text{ m/s (3 sig. fig.)}$$

9. **Topic: Coordinate Geometry (Circles)**

Given $x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

(i) Equation of the circle: $(x - 2)^2 + (y + 1)^2 = 5^2$

$$x^2 - 4x + 4 + y^2 + 2y + 1 - 25 = 0$$

$$x^2 - 4x + y^2 + 2y - 20 = 0$$

$$x^2 + y^2 - 4x + 2y - 20 = 0$$
 (2)

Comparing coefficients between (1) and (2)

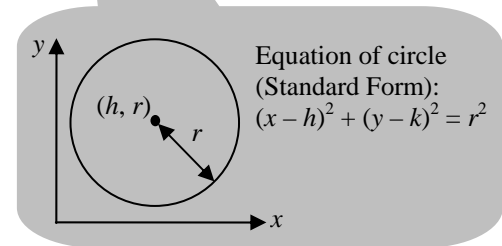
$$\Rightarrow -4 = 2g$$

$$g = -2$$

$$\Rightarrow 2 = 2f$$

$$f = 1$$

$$\Rightarrow c = -20$$



(ii) Since AC is parallel to x-axis \Rightarrow y-coordinate of A = -1

$$\text{Sub } y = -1 \text{ into (2), } x^2 + (-1)^2 - 4x + 2(-1) - 20 = 0$$

$$x^2 - 4x - 21 = 0$$

$$(x-7)(x+3) = 0$$

$$x - 7 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 7 \text{ (rejected)} \quad \quad \quad x = -3$$

\therefore Coordinates of A are (-3, -1)

(iii) Gradient of AB = Gradient of OA

$$= \frac{-1-0}{-3-0}$$

$$= \frac{1}{3}$$

$$\text{Equation of AB: } y + 1 = \frac{1}{3}(x + 3)$$

$$y = \frac{1}{3}x \dots\dots\dots (3)$$

Equation of line with gradient m and point (x, y) :
 $y - y_1 = m(x - x_1)$

(iv) Sub (2) into (1),

$$x^2 + \left(\frac{1}{3}x\right)^2 - 4x + 2\left(\frac{1}{3}x\right) - 20 = 0$$

$$x^2 + \frac{1}{9}x^2 - 4x + \frac{2}{3}x - 20 = 0$$

$$9x^2 + x^2 - 36x + 6x - 180 = 0$$

$$10x^2 - 30x - 180 = 0$$

$$x^2 - 3x - 18 = 0$$

$$(x-6)(x+3) = 0$$

$$x - 6 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 6 \quad \quad \quad x = -3 \text{ (rejected)}$$

$$\text{Sub } x = 6 \text{ into (2), } y = \frac{1}{3}(6)$$

$$= 2$$

\therefore Coordinates of B are (6, 2)

10. Topics: Applications of Differentiation (Stationary Points)

$$\frac{d^2y}{dx^2} = 6x - 6$$

$$\frac{dy}{dx} = \int (6x - 6) dx$$

$$= 3x^2 - 6x + c$$

$$\text{When } x = 3, \frac{dy}{dx} = 12$$

$$\Rightarrow 12 = 3(9) - 6(3) + c$$

$$c = 3$$

$$\therefore \frac{dy}{dx} = 3x^2 - 6x + 3$$

$$y = \int 3x^2 - 6x + 3 dx$$

$$= x^3 - 3x^2 + 3x + d$$

Sub (3, 10)

$$\Rightarrow 10 = 27 - 27 + 9 + d$$

$$d = 1$$

$$\therefore y = x^3 - 3x^2 + 3x + 1 \dots\dots\dots (1)$$

For stationary point on the curve: $\frac{dy}{dx} = 0$

$$3x^2 - 6x + 3 = 0$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$x = 1$$

$$\text{Sub } x = 1 \text{ into (1), } y = 1 - 3 + 3 + 1$$

$$= 2$$

\therefore Coordinates of the stationary point are (1, 2)

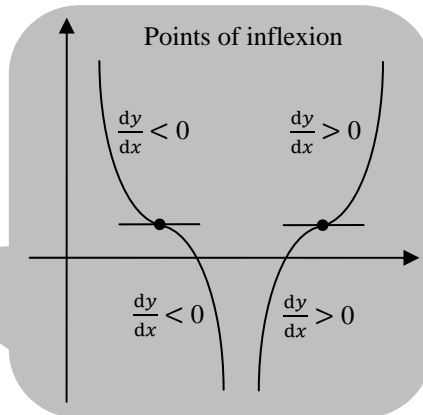
For (1, 2), using the 1st derivative test

x	0.9	1	1.1
$\frac{dy}{dx}$	+	0	+
	/	—	/

When $x = 0.9$, $\frac{dy}{dx} = 3(0.9)^2 - 6(0.9) + 3$
 $= 0.03 > 0$

When $x = 1.1$, $\frac{dy}{dx} = 3(1.1)^2 - 6(1.1) + 3$
 $= 0.03 > 0$

$\therefore (1, 2)$ is a point of inflexion.



11. Topics: Further Trigonometric Identities (R-formula)

(i) Given $\angle COD = \theta$

$$\begin{aligned} \angle AOB &= \angle AOD - \angle COD \\ &= 90^\circ - \theta \\ \angle OAB &= 90^\circ - \angle AOB \\ &= 90^\circ - (90^\circ - \theta) \\ &= \theta \\ \angle BAO &= \theta \\ \cos \theta &= \frac{AB}{17} \\ AB &= 17\cos \theta \\ \sin \theta &= \frac{OB}{17} \\ OB &= 17\sin \theta \\ \cos \theta &= \frac{OC}{31} \\ OC &= 31\cos \theta \\ \therefore BC &= OC - OB \\ &= 31\cos \theta - 17\sin \theta \\ \sin \theta &= \frac{CD}{31} \\ CD &= 31\sin \theta \\ \therefore AB + BC + CD &= 17\cos \theta + 31\cos \theta - 17\sin \theta + 31\sin \theta \\ &= (48\cos \theta + 14\sin \theta) \text{ cm (Shown)} \end{aligned}$$

(ii) Given $AB + BC + CD = 49$ Proved in part (i)
 $\Rightarrow 48\cos\theta + 14\sin\theta = 49 \dots\dots\dots (1)$

Using R-formula,

$$48\cos\theta + 14\sin\theta = R\cos(\theta - \alpha)$$

$$= R[\cos\theta\cos\alpha + \sin\theta\sin\alpha]$$

$$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

R-Formula:
 $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$
 where
 $\tan\alpha = \frac{b}{a}$
 $R = \sqrt{a^2 + b^2}$

Comparing coefficients, $48 = R\cos\alpha \dots\dots\dots (1)$
 $14 = R\sin\alpha \dots\dots\dots (2)$

$$\frac{(2)}{(1)}: \frac{R\sin\alpha}{R\cos\alpha} = \frac{14}{48}$$

$$\tan\alpha = 0.29166$$

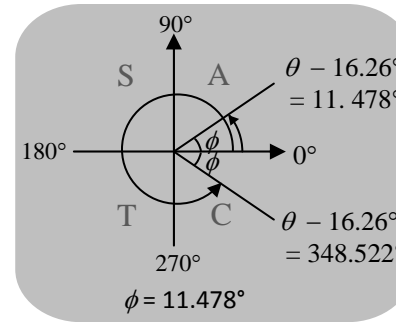
$$\alpha = 16.26^\circ$$

$(1)^2 + (2)^2: R^2\cos^2\alpha + R^2\sin^2\alpha = 48^2 + 14^2$
 $R^2(\cos^2\alpha + \sin^2\alpha) = 2500$

$\cos^2\theta + \sin^2\theta = 1$
 $R^2 = 2500$
 $R = 50 \text{ or } -50 \text{ (rejected)}$

$\therefore 48\cos\theta + 14\sin\theta = 50\cos(\theta - 16.26^\circ) \dots\dots\dots (2)$

Sub (2) into (1), $50\cos(\theta - 16.26^\circ) = 49$
 $\cos(\theta - 16.26^\circ) = 0.98$
 Basic $\angle \phi = 11.478^\circ$
 $\theta - 16.26^\circ = 11.478^\circ, 360^\circ - 11.478^\circ$



$\theta = 23.73^\circ, 364.78^\circ$
 $= 27.73^\circ, 4.78^\circ$
 $\approx 4.8^\circ, 27.7^\circ \text{ (1 d.p.)}$

(iii) Maximum value of $AB + BC + CD = 50$
 When $\cos(\theta - 16.26^\circ) = 1$
 $\Rightarrow \theta - 16.26^\circ = 0^\circ$

Max/min values of $R\cos(\theta - \alpha) = \pm R$

$\theta = 16.26^\circ$
 $\approx 16.3^\circ \text{ (1 d.p.)}$