

**ADDITIONAL MATHEMATICS**  
Paper 1 Suggested Solutions

**4038/01**  
**October/November 2009**

1. **Topic: Polynomials**

$$\text{Let } f(x) = 2x^3 + ax^2 + bx + 3$$

$$f(1) = 0$$

$$\Rightarrow 2(1)^3 + a(1)^2 + b(1) + 3 = 0$$

$$2 + a + b + 3 = 0$$

$$a = -5 - b \quad - (1)$$

$$f(-2) = 15$$

$$\Rightarrow 2(-2)^3 + a(-2)^2 + b(-2) + 3 = 15$$

$$-16 + 4a - 2b + 3 = 15$$

$$4a - 2b = 28$$

$$2a - b = 14 \quad - (2)$$

Sub (1) into (2),

$$2(-5 - b) - b = 14$$

$$-10 - 2b - b = 14$$

$$-3b = 24$$

$$b = -8$$

Sub  $b = -8$  into (1),

$$a = -5 + 8$$

$$= 3$$

$\therefore a = 3$  and  $b = -8$

**Factor Theorem:**  
 $f(a) = 0 \Leftrightarrow (x - a)$  is  
a factor of  $f(x)$

**Remainder Theorem:**  
 $f(x)$  divided by  $(x - a)$   
 $\Rightarrow$  remainder is  $f(a)$

2. **Topics: Applications of Differentiation**  
**(Increasing and Decreasing Functions)**

$$y = \frac{\ln x}{x}$$

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

**Quotient rule:**

For  $y = \frac{u}{v}$ ,

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Since  $y$  is an increasing function of  $x$ ,  $\frac{dy}{dx} > 0$

$$\Rightarrow 1 - \ln x > 0 \text{ as } x^2 > 0$$

$$-\ln x > -1$$

$$\ln x < 1$$

$$x < e^1$$

$\therefore$  The set of values of  $x = \{x: 0 < x < e, x \in \mathbb{R}\}$

3. **Topic: Surds**

$$x\sqrt{24} = x\sqrt{3} + \sqrt{6}$$

$$x\sqrt{24} - x\sqrt{3} = \sqrt{6}$$

$$x(\sqrt{24} - \sqrt{3}) = \sqrt{6}$$

$$x = \frac{\sqrt{6}}{(\sqrt{24} - \sqrt{3})} \times \frac{(\sqrt{24} + \sqrt{3})}{(\sqrt{24} + \sqrt{3})}$$

$$= \frac{\sqrt{6}\sqrt{24} + \sqrt{6}\sqrt{3}}{24 - 3}$$

$$= \frac{\sqrt{144} + \sqrt{18}}{21}$$

$$= \frac{12 + \sqrt{9 \times 2}}{21}$$

$$= \frac{12 + 3\sqrt{2}}{21}$$

$$= \frac{4 + \sqrt{2}}{7}$$

**Rationalising the denominator:**

$$\frac{1}{\sqrt{a} + \sqrt{b}} \times \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{a - b}$$

$\therefore a = 4, b = 2$

#### 4. Topic: Logarithms

(i)  $\lg(x+14) - \lg(x-2) = 2\lg 5$

$$\lg\left(\frac{x+14}{x-2}\right) = \lg 5^2$$

$$n \log_a x = \log_a x^n$$

$$\frac{x+14}{x-2} = 25$$

$$x+14 = 25(x-2)$$

$$x+14 = 25x-50$$

$$24x = 64$$

$$x = 2\frac{2}{3}$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

(ii)  $\log_2 y + \log_4 y = 6$

$$\log_2 y + \frac{\log_2 y}{\log_2 4} = 6$$

$$\log_2 y + \frac{\log_2 y}{2} = 6$$

$$2\log_2 y + \log_2 y = 12$$

$$3\log_2 y = 12$$

$$\log_2 y = 4$$

$$y = 2^4$$

$$= 16$$

Changing of base:

$$\log_a N = \frac{\log_b N}{\log_b a}$$

#### 5. Topics: Applications of Differentiation (Gradients, Tangents & Normal)

$$y = 1 - 3\tan x$$

When curve intersects y-axis,  $x = 0$ ,

$$y = 1 - 3\tan 0$$

$$= 1$$

$$\frac{dy}{dx} = -3\sec^2 x$$

When  $x = 0$ ,  $\frac{dy}{dx} = -3\sec^2 0$

$$= -3$$

$$\therefore \text{Gradient of tangent} = -3$$

$$\text{Gradient of normal} = \frac{1}{3}$$

Equation of normal:  $y - 1 = \frac{1}{3}(x - 0)$

$$y = \frac{1}{3}x + 1 \quad \text{--- (1)}$$

Sub  $(k, 3)$  into (1),  $3 = \frac{1}{3}k + 1$

$$\frac{1}{3}k = 2$$

$$k = 6$$

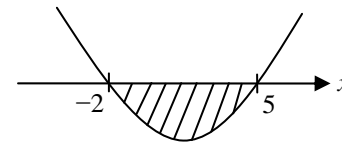
#### 6. Topic: Quadratic Functions and Inequalities

(i)  $y = 2x^2 - 6x + c$

When  $c = -20$ ,  $2x^2 - 6x - 20 \leq 0$

$$x^2 - 3x - 10 \leq 0$$

$$(x-5)(x+2) \leq 0$$



The set of values of  $x = \{x: -2 \leq x \leq 5, x \in \mathbb{R}\}$

(ii)  $y = 2x^2 - 6x + c$  — (1)  
 $y + 2x = 8$  — (2)

Sub (1) into (2),  
 $2x^2 - 6x + c + 2x = 8$   
 $2x^2 - 4x + c - 8 = 0$

$a = 2, b = -4, c = c - 8$   
 Since the line is a tangent to the curve,  $b^2 - 4ac = 0$   
 $(-4)^2 - 4(2)(c - 8) = 0$   
 $16 - 8c + 64 = 0$   
 $8c = 80$   
 $c = 10$

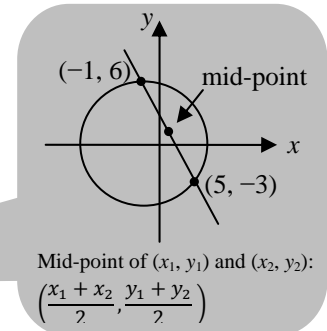
7. **Topic: Coordinate Geometry**

$x^2 + 2y^2 + 5x = 68$  — (1)  
 $2y + 3x = 9$   
 $2y = 9 - 3x$   
 $y = \frac{9-3x}{2}$  — (2)

Sub (2) into (1),  
 $x^2 + 2\left(\frac{9-3x}{2}\right)^2 + 5x = 68$   
 $x^2 + \frac{2(9-3x)^2}{2} + 5x = 68$   
 $x^2 + \frac{81-54x+9x^2}{2} + 5x = 68$   
 $2x^2 + 81 - 54x + 9x^2 + 10x = 136$   
 $11x^2 - 44x - 55 = 0$   
 $x^2 - 4x - 5 = 0$   
 $(x + 1)(x - 5) = 0$   
 $x = -1$  or  $x = 5$

Sub  $x = -1$  into (2),  $y = \frac{9-3(-1)}{2} = 6$   
 Sub  $x = 5$  into (2),  $y = \frac{9-3(5)}{2} = -3$

The coordinates of intersection points  
 $\Rightarrow (-1, 6)$  and  $(5, -3)$   
 $\therefore$  Mid-point =  $\left(\frac{-1+5}{2}, \frac{6-3}{2}\right) = (2, 1.5)$



8. **Topic: Further Trigonometric Identities (Factor and Double Angle formula)**

(i)  $\cos 3x - \cos x = -4\sin^2 x \cos x$

L.H.S. =  $-2\sin\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)$   
 $= -2\sin 2x \sin x$   
 $= -2(2\sin x \cos x)\sin x$   
 $= -4\sin^2 x \cos x$   
 $= \text{R.H.S. (Shown)}$

Factor Formula:  
 $\cos A + \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

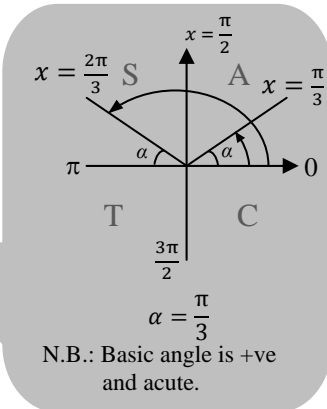
Double Angle Formula:  
 $\sin 2A = 2\sin A \cos A$

(ii)  $\cos 3x + 2\cos x = 0$   
 $\cos 3x - \cos x = -3\cos x$   
 $-4\sin^2 x \cos x = -3\cos x$   
 $4\sin^2 x \cos x - 3\cos x = 0$   
 $\cos x(4\sin^2 x - 3) = 0$   
 $\cos x = 0$  or  $4\sin^2 x - 3 = 0$   
 $x = \frac{\pi}{2}$   $\sin^2 x = \frac{3}{4}$

Using proof in (i)

$\sin x = \pm\sqrt{\frac{3}{4}}$   
 Basic angle  $\alpha = \frac{\pi}{3}$   
 $x = \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$



## 9. Topic: Trigonometric Functions

$$f(x) = 3\sin\left(\frac{x}{3}\right) - 1$$

$$(i) \quad -1 \leq \sin\left(\frac{x}{3}\right) \leq 1$$

$$-3 \leq 3\sin\left(\frac{x}{3}\right) \leq 3$$

$$-3 - 1 \leq 3\sin\left(\frac{x}{3}\right) - 1 \leq 3 - 1$$

$$-4 \leq 3\sin\left(\frac{x}{3}\right) - 1 \leq 2$$

Maximum value of  $f(x) = 2$

Minimum value of  $f(x) = -4$

The values of  $a \sin bx$  lie between  $a$  and  $-a$ .

(ii) Amplitude of  $f = 3$

(iii) Period of  $f = \frac{360^\circ}{\frac{1}{3}}$   
 $= 1080^\circ$

(iv)  $3\sin\left(\frac{x}{3}\right) - 1 = 0$

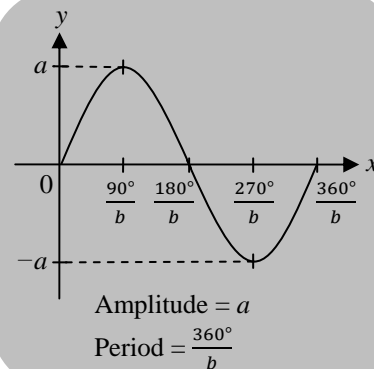
$$\sin\left(\frac{x}{3}\right) = \frac{1}{3}$$

Basic  $\angle = 19.47^\circ$

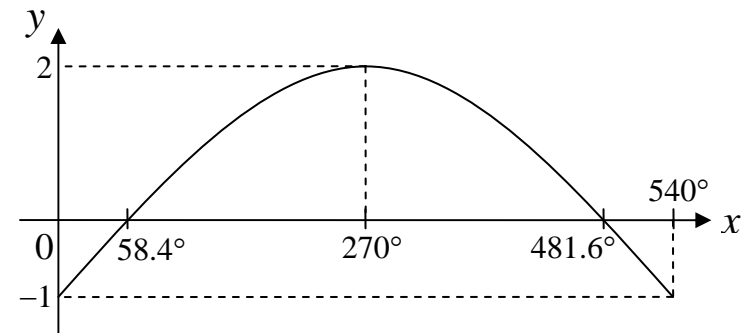
$$\frac{x}{3} = 19.47^\circ$$

$$x = 58.41^\circ$$

$$\approx 58.4^\circ \text{ (1 d.p.)}$$



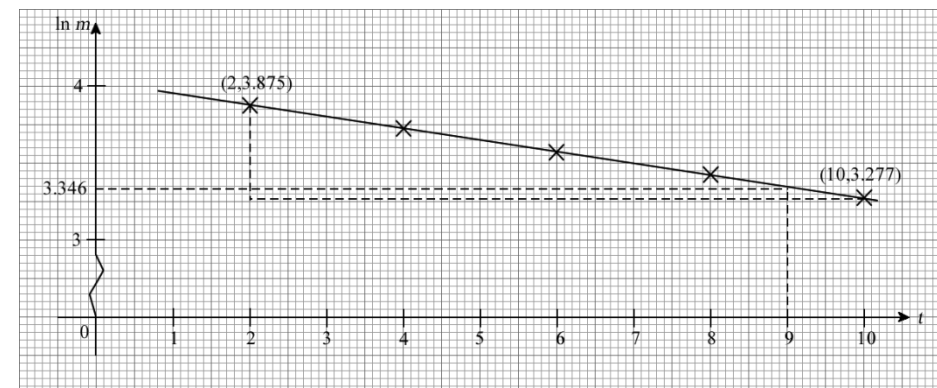
(v)  $y = 3\sin\left(\frac{x}{3}\right) - 1$



## 10. Topic: Straight Line Graphs/ Linear Law

(i)

$t$	2	4	6	8	10
$\ln m$	3.875	3.725	3.575	3.424	3.277



$$\begin{aligned}
 \text{(ii)} \quad m &= m_0 e^{-kt} \\
 \ln m &= \ln m_0 e^{-kt} \\
 &= \ln m_0 + \ln e^{-kt} \\
 &= \ln m_0 - kt \ln e \\
 \ln m &= \ln m_0 - kt \\
 \Rightarrow \text{Gradient} &= -k \\
 \frac{3.277 - 3.875}{10 - 2} &= -k \\
 k &= 0.07475 \\
 &\approx 0.0748 \text{ (3 s.f.)} \\
 \Rightarrow \ln m_0 &= 4.04 \text{ (intercept on } \ln m \text{ axis)} \\
 m_0 &= e^{4.04} \\
 &= 56.82 \\
 &\approx \mathbf{56.8 \text{ (3 s.f.)}} \\
 \text{(iii) When } m &= \frac{1}{2} m_0, \ln m = \ln \left( \frac{56.82}{2} \right) \\
 &\approx 3.346
 \end{aligned}$$

From graph,  $t = \mathbf{9 \text{ hours}}$

## 11. Topic: Coordinate Geometry

$$\begin{aligned}
 \text{(i)} \quad \text{Gradient of } AD &= \frac{6+4}{0-2} \\
 &= -4
 \end{aligned}$$

$$\text{Gradient of } AB = \frac{1}{4}$$

$$\text{Equation of } AB: y - 6 = \frac{1}{4}(x - 0)$$

$$y = \frac{1}{4}x + 6 \quad \text{--- (1)}$$

$$\text{(ii)} \quad y = x \quad \text{--- (2)}$$

Sub (1) into (2),

$$x = \frac{1}{4}x + 6$$

$$4x = x + 24$$

$$3x = 24$$

$$x = 8$$

$$\therefore y = 8$$

**Coordinates of B are (8, 8).**

(iii) Let C be (x, y),

$$\text{Length of } DC = 2 \times \text{Length of } AB$$

$$\sqrt{(2-x)^2 + (-2-y)^2} = 2\sqrt{(0-8)^2 + (6-8)^2}$$

$$(2-x)^2 + (-2-y)^2 = 4(68)$$

$$(2-x)^2 + (-2-y)^2 = 272 \quad \text{--- (3)}$$

$$\text{Equation of } CD: y + 2 = \frac{1}{4}(x - 2)$$

$$y = \frac{1}{4}x - \frac{5}{2} \quad \text{--- (4)}$$

Equation of line with gradient  $m$  and point  $(x_1, y_1)$ :  
 $(y - y_1) = m(x - x_1)$

Length of line segment  
 $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Sub (4) into (3),

$$4 - 4x + x^2 + 4 + 4\left(\frac{1}{4}x - \frac{5}{2}\right) + \left(\frac{1}{4}x - \frac{5}{2}\right)^2 = 272$$

$$4 - 4x + x^2 + 4 + x - 10 + \frac{1}{16}x^2 - \frac{5}{4}x + \frac{25}{16} - 272 = 0$$

$$\frac{17}{16}x^2 - \frac{17}{4}x - \frac{1071}{4} = 0$$

$$17x^2 - 68x - 4284 = 0$$

$$x^2 - 4x - 252 = 0$$

$$(x-18)(x+14) = 0$$

$$x - 18 = 0 \quad \text{or} \quad x + 14 = 0$$

$$x = 18$$

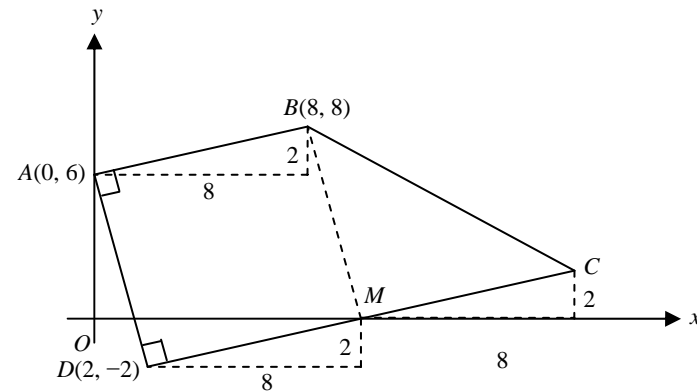
$$x = -14 \text{ (rej.)}$$

Sub  $x = 18$  into (4),  $y = \frac{1}{4}(18) - \frac{5}{2}$

$$= 2$$

$\therefore$  Coordinates of  $C$  are **(18, 2)**

**Alternate Method**



Given  $DC$  is twice the length of  $AB$ .

Let  $M$  be the mid-point of  $DC$ .

Then  $DM = AB$  and  $DM \parallel AB$

Let  $M$  be  $(a, b)$  and  $C(x, y)$

$$\text{Then } b = -2 + 2 = 0$$

$$a = 2 + 8 = 10$$

$$\therefore M(10, 10)$$

Since  $M$  is the mid-point of  $DC$ ,

$$\frac{2+x}{2} = 10 \text{ and } \frac{y+(-2)}{2} = 0$$

$\therefore$  Coordinates of  $C$  are **(18, 2)**

Mid-point of  $(x_1, y_1)$  and  $(x_2, y_2)$ :

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\begin{aligned}
 \text{(iv) Area of trapezium } ABCD &= \frac{1}{2} \begin{vmatrix} 0 & 2 & 18 & 8 & 0 \\ 6 & -2 & 2 & 8 & 6 \end{vmatrix} \\
 &= \frac{1}{2} |0 + 4 + 144 + 48 - 12 + 36 - 16 - 0| \\
 &= \frac{1}{2} |204| \\
 &= 102 \text{ units}^2
 \end{aligned}$$

Area of quadrilateral  $ABCD$  ('Shoelace Method')

$$\begin{aligned}
 &= \frac{1}{2} \begin{vmatrix} x_A & x_D & x_C & x_B & x_A \\ y_A & y_D & y_C & y_B & y_A \end{vmatrix} \\
 &= \frac{1}{2} (x_A y_D + x_D y_C + x_C y_B + x_B y_A - x_D y_A - x_C y_D - x_B y_C - x_A y_B)
 \end{aligned}$$

Note: Coordinates must be taken in an anticlockwise direction.

## 12. Topic: Differentiation and Integration

$$\text{(i) } y = (2x - 1)\sqrt{4x + 1}$$

$$\begin{aligned}
 \frac{dy}{dx} &= (2x - 1) \frac{1}{2} (4x + 1)^{-\frac{1}{2}} (4) + \sqrt{4x + 1} (2) \\
 &= \frac{2(2x-1)}{\sqrt{4x+1}} + \frac{2\sqrt{4x+1}}{1} \\
 &= \frac{2(2x-1) + 2(4x+1)}{\sqrt{4x+1}} \\
 &= \frac{4x - 2 + 8x + 2}{\sqrt{4x+1}} \\
 &= \frac{12x}{\sqrt{4x+1}}
 \end{aligned}$$

Product Rule:

For  $y = uv$ ,

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$\begin{aligned}
 \text{(ii) } \frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\
 2 &= \frac{12(2)}{\sqrt{8+1}} \times \frac{dx}{dt} \\
 2 &= \frac{24}{3} \times \frac{dx}{dt} \\
 \frac{dx}{dt} &= \frac{1}{4} \text{ units per second}
 \end{aligned}$$

**\*Hence question:**

Using answer from part (i)

$$\begin{aligned}
 \text{(iii) } \int_0^2 \frac{3x}{\sqrt{4x+1}} dx &= \frac{1}{4} \int_0^2 \frac{12x}{\sqrt{4x+1}} dx \\
 &= \frac{1}{4} [(2x - 1)\sqrt{4x + 1}]_0^2 \\
 &= \frac{1}{4} [3\sqrt{9} - (-1)(1)] \\
 &= \frac{1}{4} [9 + 1] \\
 &= 2.5
 \end{aligned}$$