

ELEMENTARY MATHEMATICS
 Paper 2 Suggested Solutions

4017/02
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1. **Topics: Algebraic Manipulation, Solutions to Quadratic Equations**

$$\begin{aligned} \text{(a)} \quad \frac{7p^2-28}{p^2+2p} &= \frac{7(p^2-4)}{p(p+2)} \\ &= \frac{7(p+2)(p-2)}{p(p+2)} \\ &= \frac{7(p-2)}{p} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 1 - \frac{3f-g}{f+2g} &= \frac{f+2g-(3f-g)}{f+2g} \\ &= \frac{f+2g-3f+g}{f+2g} \\ &= \frac{3g-2f}{f+2g} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \text{(i)} \quad x^2 + 11x - 15 &= x^2 + 11x + \left(\frac{11}{2}\right)^2 - \left(\frac{11}{2}\right)^2 - 15 \\ &= \left(x + \frac{11}{2}\right)^2 - 45.25 \\ &= (x + 5.5)^2 - 45.25 \end{aligned}$$

$$\text{(ii)} \quad x^2 + 11x - 15 = 0$$

$$\text{Sub } a = 1, b = 11, c = -15 \text{ into } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

$$\begin{aligned} x &= \frac{-11 \pm \sqrt{11^2 - 4(1)(-15)}}{2(1)} \\ &= \frac{-11 \pm \sqrt{120 + 60}}{2} \\ &= \frac{-11 \pm \sqrt{180}}{2} \\ &= \frac{-11 \pm \sqrt{181}}{2} \\ &= \mathbf{1.23, -12.23} \end{aligned}$$

Completing the Square:

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \end{aligned}$$

2. **Topic: Trigonometry**

$$\text{(a)} \quad \cos 49^\circ = \frac{AD}{9.4}$$

$$AD \approx \mathbf{6.17 \text{ m (3 sig. fig.)}}$$

$$\text{(b)} \quad \angle PAB = 90^\circ - 49^\circ - 32^\circ = 9^\circ$$

$$\sin 9^\circ = \frac{PB}{12.1}$$

$$PB \approx \mathbf{1.89 \text{ m (3 sig. fig.)}}$$

$$\begin{aligned} \text{(c)} \quad \text{Area of } \triangle APQ &= \frac{1}{2} (9.4) (12.1) \sin 32^\circ \\ &\approx \mathbf{30.1 \text{ m}^2 \text{ (3 sig. fig.)}} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \text{Using cosine rule, } PQ^2 &= 9.4^2 + 12.1^2 - 2(9.4)(12.1) \cos 32^\circ \\ PQ &\approx \mathbf{6.47 \text{ m (3 sig. fig.)}} \end{aligned}$$

3. **Topic: Arithmetic (Application of Mathematics in Practical Situations)**

$$\begin{aligned} \text{(a)} \quad \text{Total cost of making 25 000 souvenirs} &= 25000 \times \$0.90 \\ &= \mathbf{\$22500} \end{aligned}$$

$$\text{(b)} \quad \text{Cost of materials per souvenir} = \$ \left[\frac{0.9}{15} \times 5 \right] = \mathbf{\$0.30}$$

$$\text{Cost of wages per souvenir} = \$ \left[\frac{0.9}{15} \times 4 \right] = \mathbf{\$0.24}$$

$$\text{(c)} \quad \text{Total no. of hours spent} = 7 \times 5 = 35 \text{ hours}$$

$$\Rightarrow \text{Salary for 35 hours} = \$630 \text{ (Given)}$$

$$\text{Salary per hour} = \$630 \div 35 = \$18$$

$$\therefore \text{no. of souvenirs John made in one hour} = \frac{18}{0.24} = \mathbf{75}$$

$$\text{(d)} \quad \text{From (b): Original cost of materials} = \$0.30$$

$$\Rightarrow \text{Increase in cost of materials} = \$0.30 \times 50\% = \$0.15$$

$$\text{Original wages} = \$0.24$$

$$\Rightarrow \text{Increase in wages} = \$0.24 \times 10\% = \$0.024$$

$$\begin{aligned} \% \text{ increase in cost of making a souvenir} &= \frac{\$0.15 + \$0.024}{\$0.90} \times 100\% \\ &\approx \mathbf{19.3\% \text{ (3 sig. fig.)}} \end{aligned}$$

$$\text{(e)} \quad 125\% \text{ of cost price} = \$2.00$$

$$\Rightarrow \text{Cost price} = \frac{\$2.00}{\frac{125}{100}} = \mathbf{\$1.60}$$

4. **Topic: Number Patterns**

(a) $u_5 = 2^4 + 9 = 25$ (shown)

(b) $u_6 = 2^5 + 11 = 43$

(c) $u_n = 2^{n-1} + 2n - 1$

(d) $u_{20} = 2^{19} + 2(20) - 1 = 524327$

(e) (i) L.H.S.: $2^{n-1} - 2^{n-2} = \frac{2^n}{2} - \frac{2^n}{2^2}$
 $= 2^n \left(\frac{1}{2} - \frac{1}{4} \right)$
 $= 2^n \left(\frac{1}{4} \right)$
 $= 2^n (2^{-2})$
 $= 2^{n-2} = \text{R.H.S. (Shown)}$

(ii) $u_n - u_{n-1} = [2^{n-1} + 2n - 1] - [2^{n-2} + 2(n-1) - 1]$
 $= [2^{n-1} + 2n - 1] - [2^{n-2} + 2n - 3]$
 $= 2^{n-1} + 2n - 1 - 2^{n-2} - 2n + 3$
 $= 2^{n-1} - 2^{n-2} + 2$
 $= 2^{n-2} + 2$

Sub $2^{n-1} - 2^{n-2} = 2^{n-2}$
as proven in (e)(i).

5. **Topic: Algebraic Representation & Formulae**

(a) Cost of each apple = $\frac{\$12}{m} = \left(\frac{1200}{m}\right)\text{¢}$

\$12 = 1200¢
→ question requires this to
be expressed in cents.

(b) No. of remaining apples = $(m - 3)$

Selling price of each apple = $\left(\frac{1200}{m} + 2\right)\text{¢}$ (1)

∴ Total sum received from the sale of the apples

$= (m - 3)\left(\frac{1200}{m} + 2\right)$

(c) $(m - 3)\left(\frac{1200}{m} + 2\right) - 1200 = 96$

$(m - 3)\left(\frac{1200 + 2m}{m}\right) = 1296$

$(m - 3)(1200 + 2m) = 1296m$

$1200m + 2m^2 - 3600 - 6m = 1296m$

$2m^2 - 102m - 3600 = 0$

$m^2 - 51m - 1800 = 0$ (Shown)

Total cost = \$12 = 1200¢ (given)
Total sales = $(m-3)\left(\frac{1200}{m} + 2\right)$ from (b)
Profit = total sales - total cost

(d) $m^2 - 51m - 1800 = 0$

$m = \frac{51 \pm \sqrt{(-51)^2 - 4(1)(-1800)}}{2}$
 $= \frac{51 \pm \sqrt{9801}}{2}$
 $= 75 \text{ or } -24$

(e) $m = 75$ ($m = -24$ rejected ∵ number of apples cannot be negative)

Sub $m = 75$ into (1): Selling price = $\left(\frac{1200}{75} + 2\right)\text{¢}$
 $= 18\text{¢} = \$0.18$

6. **Topics: Congruence & Similarity, Angles & Triangles**

(a) $\angle LAD = \angle LCB$ (angles in same segment)

$\angle LDA = \angle LBC$ (angles in same segment)

$\angle ALD = \angle CLB = 90^\circ$ (vertically opposite angles)

∴ $\triangle LAD$ and $\triangle LCB$ are similar (AAA) (Shown)

(b) (i) $\angle CNO = 90^\circ$ ($ON \perp BC$ ∵ N is midpt of BC of isosceles $\triangle OBC$)

(ii) $\angle DCB = \angle DAB = 58^\circ$ (angles in same segment)

$\angle CON = 180^\circ - \angle CNO - (\angle DCO + \angle DCB)$ (sum of \angle s in a triangle)
 $= 180^\circ - 90^\circ - (18^\circ + 58^\circ)$
 $= 14^\circ$

$\angle CNO = 90^\circ$ from (b)(i)

(iii) $\angle CBA = 180^\circ - \angle CLB - \angle DCB$ (sum of \angle s in a triangle)

$= 180^\circ - 90^\circ - 58^\circ$
 $= 32^\circ$

$\angle DCB = 58^\circ$ from (b)(ii)

(iv) $\angle CDO = \angle DCO = 18^\circ$ (base \angle s of isosceles $\triangle DCO$)

$\angle ADC = \angle CBA$ (angles in same segment)
 $= 32^\circ$

$\angle ADO = \angle ADC - \angle CDO$
 $= 32^\circ - 18^\circ$
 $= 14^\circ$

$\angle CBA = 32^\circ$ from (b)(iii)

7. Topics: Geometrical Properties of Circles, Trigonometry

- (a) (i) $\tan \angle AOC = \frac{AC}{OC}$ ($OC \perp AC \therefore OC$ is perpendicular bisector of chord AB)

$$\begin{aligned} &= \frac{40}{50} \\ \angle AOC &= 38.66^\circ \\ \therefore \angle AOB &= 2 \times \angle AOC \\ &= 2 \times 38.66^\circ \\ &\approx 77.32^\circ \\ &\approx \mathbf{77.3^\circ \text{ (3 sig. fig.)}} \end{aligned}$$

- (ii) Using Pythagoras' theorem for $\triangle OAC$,

$$\begin{aligned} AO \text{ (length of radius of sector } OAB) &= \sqrt{40^2 + 50^2} \text{ cm} \\ &= \sqrt{4100} \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{Area of window} &= \text{Area of sector } OAB - \text{Area of } \triangle OAC \\ &= \frac{1}{2} \times AO^2 \times \angle AOB - \frac{1}{2} \times AC \times OC \\ &= \frac{1}{2} (\sqrt{4100})^2 \left(\frac{77.32^\circ}{180^\circ} \times \pi \right) - \frac{1}{2} (80)(50) \\ &\approx \mathbf{766 \text{ cm}^2 \text{ (3 sig. fig.)}} \end{aligned}$$

$\angle AOB$ must be converted to radians:
 $\frac{\theta}{180^\circ} \times \pi$ radians

- (b) (i) Using cosine rule for $\triangle DEX$,
 $EX^2 = DE^2 + DX^2 - 2(DE)(DX) \cos \angle EDX$
 $= 80^2 + 80^2 - 2(80)(80) \cos 38^\circ$
 $\therefore EX \approx 52.09 \text{ cm}$
 $\approx \mathbf{52.1 \text{ cm (3 sig. fig.)}}$

- (ii) Using Pythagoras' theorem,

$$\begin{aligned} DF = DY &= \sqrt{200^2 + 80^2} \\ &= \sqrt{46400} \text{ cm} \end{aligned}$$

Using cosine rule for $\triangle FDY$,

$$\begin{aligned} FY^2 &= DF^2 + DY^2 - 2(DF)(DY) \cos \angle FDY \\ \cos \angle FDY &= \frac{(\sqrt{46400})^2 + (\sqrt{46400})^2 - (52.09)^2}{2 \sqrt{46400} \sqrt{46400}} \\ &= 0.97076 \\ \therefore \angle FDY &\approx \mathbf{13.9^\circ \text{ (3 sig. fig.)}} \end{aligned}$$

$FY = EX \approx 52.09 \text{ cm}$
from (b)(i)

8. Topic: Mensuration

- (a) (i) Using Pythagoras' theorem,

$$\begin{aligned} \text{Slant height, } s &= \sqrt{\left(\frac{0.8}{2}\right)^2 + 2^2} \text{ cm} \\ &= \sqrt{0.4^2 + 2^2} \text{ cm} \\ &\approx 2.0396 \text{ cm} \\ &\approx \mathbf{2.04 \text{ cm (3 sig. fig.)}} \end{aligned}$$

Radius of pencil = $\frac{0.8}{2}$ cm

- (ii) Total surface area

$$\begin{aligned} &= \text{Area of cone} + \text{area of cylinder} + \text{area of circular base} \\ &= \pi(0.4)(2.0396) + 2\pi(0.4)(16) + \pi(0.4)^2 \\ &\approx \mathbf{43.3 \text{ cm}^2 \text{ (3 sig. fig.)}} \end{aligned}$$

Sub $s = 2.0396$
from (a)(i) into
area of cone = πrs .

- (b) Volume of pencil

$$\begin{aligned} &= \text{Volume of cone} + \text{volume of cylinder} \\ &= \frac{1}{3} \pi (0.4)^2 (2) + \pi (0.4)^2 (16) \\ &\approx 8.378 \text{ cm}^3 \\ &\approx \mathbf{8.38 \text{ cm}^3 \text{ (3 sig. fig.)}} \end{aligned}$$

- (c) (i) Width of box = $6 \times$ pencil diameter = $6 \times 0.8 \text{ cm} = 4.8 \text{ cm}$
Height of box = $2 \times$ pencil diameter = $2 \times 0.8 \text{ cm} = 1.6 \text{ cm}$
Length of box = $1 \times$ pencil length = $(16+2) \text{ cm} = 18 \text{ cm}$
 \therefore volume of box = $4.8 \text{ cm} \times 1.6 \text{ cm} \times 18 \text{ cm}$
 $= \mathbf{138.24 \text{ cm}^3 \text{ (Shown)}}$

- (ii) Volume of box not occupied by the pencils

$$\begin{aligned} &= \text{Volume of box} - \text{total volume of 12 pencils in box} \\ &= 138.24 \text{ cm}^3 - 12 \times 8.378 \text{ cm}^3 \\ &= 37.704 \text{ cm}^3 \end{aligned}$$

Volume of each pencil
 $\approx 8.378 \text{ cm}^3$ from (b)

$$\begin{aligned} \therefore \% \text{ of the volume not occupied by the pencils} &= \frac{37.704}{138.24} \times 100\% \\ &\approx \mathbf{27.3 \%} \end{aligned}$$

9. Topics: Vectors in Two Dimensions, Probability

(a) (i) $BC = \sqrt{(5-7)^2 + (4-9)^2}$ units
 $= \sqrt{29}$ units
 ≈ 5.39 units (3 sig. fig.)

Sub $B(5, 4), C(7, 9)$ into
 $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

(ii) Gradient of $AB, m = \frac{4-1}{5-(-3)} = \frac{3}{8}$

(iii) Sub $m = \frac{3}{8}$ into $y = mx + c$:
 $y = \frac{3}{8}x + c \dots\dots\dots (1)$

Sub $A(-3, 1), B(5, 4)$ into
 $m = \frac{y_2 - y_1}{x_2 - x_1}$

Sub $A(-3, 1)$ into (1): $1 = \frac{3}{8}(-3) + c$
 $c = \frac{17}{8}$

$\therefore y = \frac{3}{8}x + \frac{17}{8}$

(iv) Mid-point of $AC = \left(\frac{-3+7}{2}, \frac{1+9}{2}\right)$
 $\therefore E = (2, 5)$

Sub $A(-3, 1), C(7, 9)$
 into $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

(v) Let the coordinates of $D = (x, y)$

Since $ABCD$ are the vertices of a parallelogram,
 \Rightarrow Mid-point of $BD =$ Mid-point of $AC = E$

Hence, mid-point of $BD: \left(\frac{x+5}{2}, \frac{y+4}{2}\right) = (2, 5)$

Coordinates of E from (a)(iv)

$\Rightarrow \frac{x+5}{2} = 2 \quad \frac{y+4}{2} = 5$
 $x+5=4 \quad y=10-4$
 $x=-1 \quad y=6$

$\Rightarrow D = (-1, 6) \Rightarrow$ Position vector $\overrightarrow{OD} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

Position vector $\overrightarrow{OE} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$\overrightarrow{ED} = \overrightarrow{OD} - \overrightarrow{OE}$
 $= \begin{pmatrix} -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix}$
 $= \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

(b) (i)

	1	2	3	4	5	6
1		1,2	1,3 [♦]	1,4	1,5	1,6 [♦]
2	2,1		2,3 [♦]	2,4 [♠]	2,5	2,6 ^{♠♠♠}
3	3,1 [♦]	3,2 [♦]		3,4 [♦]	3,5 ^{♠♦}	3,6 [♦]
4	4,1	4,2 [♠]	4,3 [♦]		4,5	4,6 ^{♠♦}
5	5,1	5,2	5,3 ^{♠♦}	5,4		5,6 [♦]
6	6,1 [♦]	6,2 ^{♠♠♠}	6,3 [♦]	6,4 ^{♠♦}	6,5 [♦]	

- ♠ - both balls even
- ♣ - sum is 8
- ♦ - at least one is multiple of 3

Without replacement \Rightarrow
 $(1, 1), (2, 2) \dots (6, 6)$
 outcomes are impossible.

From the possibility diagram, total no. of possible outcomes = 30

(ii) (a) $P(\text{both have even number}) = \frac{6}{30}$
 $= \frac{1}{5}$

$\frac{\# \text{ of } \spadesuit}{\text{Total \# of outcomes}}$
 Check: $\frac{1}{6} \times \frac{1}{5} \times 6$

(b) $P(\text{sum of numbers drawn is 8}) = \frac{4}{30}$
 $= \frac{2}{15}$

$\frac{\# \text{ of } \clubsuit}{\text{Total \# of outcomes}}$
 Check: $\frac{1}{6} \times \frac{1}{5} \times 4$

(c) $P(\text{product is 7}) = \frac{0}{30} = 0$

(d) $P(\text{at least one of the no. drawn is a multiple of 3}) = \frac{8}{30}$
 $= \frac{3}{5}$

$\frac{\# \text{ of } \blacklozenge}{\text{Total \# of outcomes}}$
 Check: $\frac{1}{6} \times \frac{1}{5} \times 18$

10. Topic: Graphical Solution of Equations

- (a) Sub $x = 4$ into $y = \frac{1}{5}x(12 - x^2)$: $p = \frac{1}{5}(4)(12 - 4^2) = -3.2$
 (b) See graph.
 (c) Plot $y = 1$ for the range $-3 \leq x \leq 4$.

From graph,

$y = \frac{1}{5}x(12 - x^2)$ intersects $y = 1$ at $x = 0.42, 3.23$

\therefore Solution of $\frac{1}{5}x(12 - x^2) = 1$: $x = 0.42, 3.23$

Check: $x^3 - 12x + 5 = 0$
 $\Rightarrow x = 3.23, -3.66, 0.42$

- (d) From graph, gradient of tangent at $(3, 1.8) = \frac{4 - (-0.5)}{2.25 - 3.75} = -3$

AMaths students:

Check: $\frac{dy}{dx} = \frac{12}{5} - \frac{3x^2}{5}$
 Sub $x = 3 \Rightarrow \frac{dy}{dx} = -3$

- (e) Since $2x + y = 2$ is linear, sub the values of $x = -1$ and $x = 3$ to obtain the y -values of the two points:

x	-1	3
y	4	-4

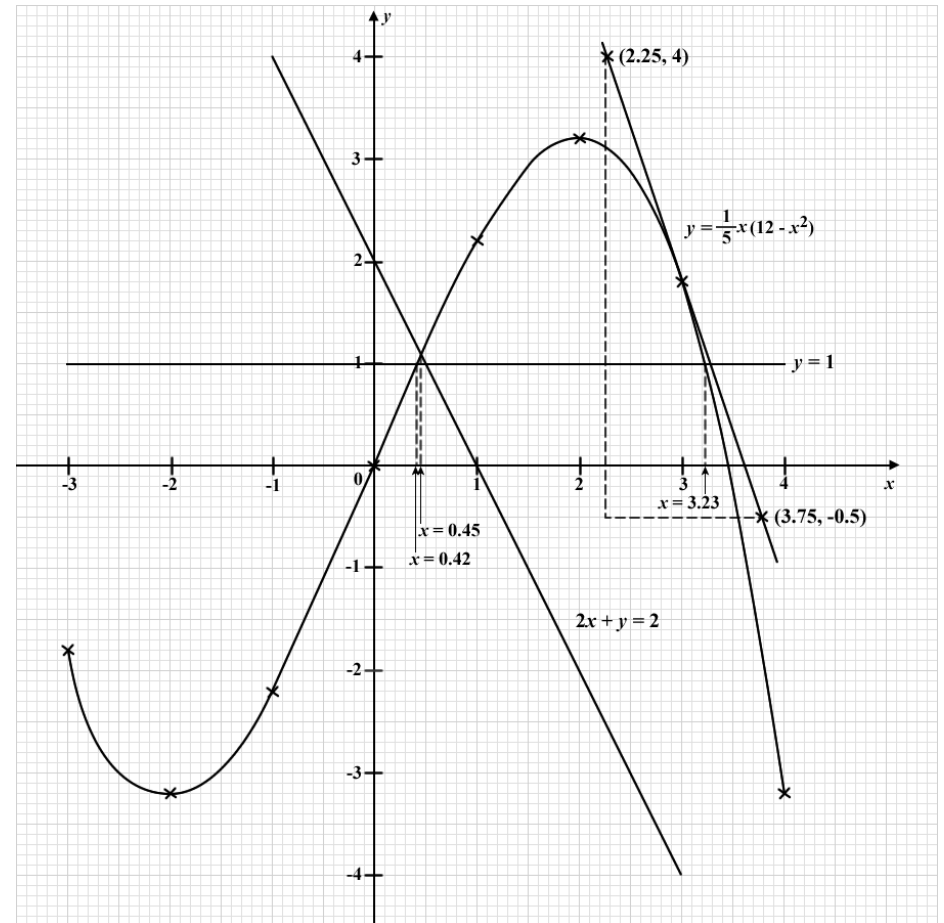
Join up these two points to get the graph of $2x + y = 2$.

- (f) (i) From graph, x -coordinate = **0.45**

(ii) $\frac{1}{5}x(12 - x^2) = 2 - 2x$
 $x^3 - 22x + 10 = 0 \dots\dots\dots (1)$

Comparing coefficients of (1) with $x^3 + Ax^2 + Bx + C = 0$,

$A = 0, B = -22, C = 10$



11. Topic: Data Analysis (Statistics, Cumulative Frequency Distribution)

(a) Mean mark = $\frac{\sum fx}{\sum f}$

$$= \frac{30(5) + 70(15) + 25(140) + 35(110) + 45(50)}{400}$$

$$= 27$$

Use the mid-value of each interval for x .

(b)

Mark (x)	$x = 0$	$x \leq 10$	$x \leq 20$	$x \leq 30$	$x \leq 40$	$x \leq 50$
No. of people	0	30	100	240	350	400

freq. ($x \leq 10$) +
freq. ($10 < x \leq 20$)
= 30 + 70

freq. ($x \leq 20$) +
freq. ($20 < x \leq 30$)
= 100 + 140

freq. ($x \leq 30$) +
freq. ($30 < x \leq 40$)
= 240 + 110

freq. ($x \leq 40$) +
freq. ($40 < x \leq 50$)
= 350 + 50

(d) From the curve in (c):

(i) Median = **27.5**

50% of 400 people scored below this mark

(ii) Upper Quartile (UQ) = **35**

(iii) Lower Quartile (LQ) = **20**

75% of 400 people scored below this mark

$$\text{Interquartile Range} = \text{UQ} - \text{LQ}$$

$$= 35 - 20$$

$$= 15$$

(e) 15% of people = $\frac{14}{100} \times 400 = 60$

15% scored *above* this mark =
85% scored *below* this mark

From the curve in (c),

Least mark needed to go on to the second round = **38.5**

(c)

