

ELEMENTARY MATHEMATICS Paper 2 Suggested Solutions

4016/02
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1. Topics: Algebraic Manipulation, Solutions to Quadratic Equations

$$\begin{aligned} \text{(a)} \quad \frac{7p^2-28}{p^2+2p} &= \frac{7(p^2-4)}{p(p+2)} \\ &= \frac{7(p+2)(p-2)}{p(p+2)} \\ &= \frac{7(p-2)}{p} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 1 - \frac{3f-g}{f+2g} &= \frac{f+2g-(3f-g)}{f+2g} \\ &= \frac{f+2g-3f+g}{f+2g} \\ &= \frac{3g-2f}{f+2g} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \text{(i)} \quad x^2 + 11x - 15 &= x^2 + 11x + \left(\frac{11}{2}\right)^2 - \left(\frac{11}{2}\right)^2 - 15 \\ &= \left(x + \frac{11}{2}\right)^2 - 45.25 \\ &= (x + 5.5)^2 - 45.25 \end{aligned}$$

$$\text{(ii)} \quad x^2 + 11x - 15 = 0$$

From (c)(ii):

$$(x + 5.5)^2 - 45.25 = 0$$

$$(x + 5.5)^2 = 45.25$$

$$x + 5.5 = \pm 6.7268$$

$$x = 1.23 \text{ or } -12.23$$

Completing the Square:

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \end{aligned}$$

Hence question (no
Otherwise stated)

⇒ you cannot use

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. Topic: Trigonometry

$$\text{(a)} \quad \cos 49^\circ = \frac{AD}{9.4}$$

$$AD \approx 6.17 \text{ m (3 sig. fig.)}$$

$$\text{(b)} \quad \angle PAB = 90^\circ - 49^\circ - 32^\circ = 9^\circ$$

$$\sin 9^\circ = \frac{PB}{12.1}$$

$$PB \approx 1.89 \text{ m (3 sig. fig.)}$$

$$\begin{aligned} \text{(c)} \quad \text{Area of } \triangle APQ &= \frac{1}{2} (9.4)(12.1) \sin 32^\circ \\ &\approx 30.1 \text{ m}^2 \text{ (3 sig. fig.)} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \text{Using cosine rule, } PQ^2 &= 9.4^2 + 12.1^2 - 2(9.4)(12.1) \cos 32^\circ \\ PQ &\approx 6.47 \text{ m (3 sig. fig.)} \end{aligned}$$

3. Topic: Arithmetic (Application of Mathematics in Practical Situations)

$$\begin{aligned} \text{(a)} \quad \text{Total cost of making 25 000 souvenirs} &= 25000 \times \$0.90 \\ &= \$22500 \end{aligned}$$

$$\text{(b)} \quad \text{Cost of materials per souvenir} = \$ \left[\frac{0.9}{15} \times 5 \right] = \$0.30$$

$$\text{Cost of wages per souvenir} = \$ \left[\frac{0.9}{15} \times 4 \right] = \$0.24$$

$$\text{(c)} \quad \text{Total no. of hours spent} = 7 \times 5 = 35 \text{ hours}$$

$$\Rightarrow \text{Salary for 35 hours} = \$630 \text{ (Given)}$$

$$\text{Salary per hour} = \$630 \div 35 = \$18$$

$$\therefore \text{no. of souvenirs John made in one hour} = \frac{18}{0.24} = 75$$

$$\text{(d)} \quad \text{From (b): Original cost of materials} = \$0.30$$

$$\Rightarrow \text{Increase in cost of materials} = \$0.30 \times 50\% = \$0.15$$

$$\text{Original wages} = \$0.24$$

$$\Rightarrow \text{Increase in wages} = \$0.24 \times 10\% = \$0.024$$

$$\begin{aligned} \% \text{ increase in cost of making a souvenir} &= \frac{\$0.15 + \$0.024}{\$0.90} \times 100\% \\ &\approx 19.3\% \text{ (3 sig. fig.)} \end{aligned}$$

$$\text{(e)} \quad 125\% \text{ of cost price} = \$2.00$$

$$\Rightarrow \text{Cost price} = \frac{\$2.00}{\frac{125}{100}} = \$1.60$$

4. **Topic: Number Patterns**

(a) $u_5 = 2^4 + 9 = 25$ (shown)

(b) $u_6 = 2^5 + 11 = 43$

(c) $u_n = 2^{n-1} + 2n - 1$

(d) $u_{20} = 2^{19} + 2(20) - 1 = 524327$

(e) (i) L.H.S.: $2^{n-1} - 2^{n-2} = \frac{2^n}{2} - \frac{2^n}{2^2}$
 $= 2^n \left(\frac{1}{2} - \frac{1}{4} \right)$
 $= 2^n \left(\frac{1}{4} \right)$
 $= 2^n (2^{-2})$
 $= 2^{n-2} = \text{R.H.S. (Shown)}$

(ii) $u_n - u_{n-1} = [2^{n-1} + 2n - 1] - [2^{n-2} + 2(n-1) - 1]$
 $= [2^{n-1} + 2n - 1] - [2^{n-2} + 2n - 3]$
 $= 2^{n-1} + 2n - 1 - 2^{n-2} - 2n + 3$
 $= 2^{n-1} - 2^{n-2} + 2$
 $= 2^{n-2} + 2$

Sub $2^{n-1} - 2^{n-2} = 2^{n-2}$
as proven in (e)(i).

5. **Topic: Algebraic Representation & Formulae**

(a) Cost of each apple = $\frac{\$12}{m} = \left(\frac{1200}{m}\right)\text{¢}$
 (b) No. of remaining apples = $(m - 3)$

Selling price of each apple = $\left(\frac{1200}{m} + 2\right)\text{¢}$ (1)

∴ Total sum received from the sale of the apples

$= (m - 3)\left(\frac{1200}{m} + 2\right)$

(c) $(m - 3)\left(\frac{1200}{m} + 2\right) - 1200 = 96$

$(m - 3)\left(\frac{1200 + 2m}{m}\right) = 1296$

$(m - 3)(1200 + 2m) = 1296m$

$1200m + 2m^2 - 3600 - 6m = 1296m$

$2m^2 - 102m - 3600 = 0$

$m^2 - 51m - 1800 = 0$ (Shown)

Total cost = \$12 = 1200¢ (given)
 Total sales = $(m - 3)\left(\frac{1200}{m} + 2\right)$ from (b)
 Profit = total sales - total cost

\$12 = 1200¢
 → question requires this to be expressed in cents.

(d) $m^2 - 51m - 1800 = 0$
 $m = \frac{51 \pm \sqrt{(-51)^2 - 4(1)(-1800)}}{2}$
 $= \frac{51 \pm \sqrt{9801}}{2}$
 $= 75 \text{ or } -24$

(e) $m = 75$ ($m = -24$ rejected ∵ number of apples cannot be negative)

Sub $m = 75$ into (1): Selling price = $\left(\frac{1200}{75} + 2\right)\text{¢}$
 $= 18\text{¢} = \$0.18$

6. **Topics: Congruence & Similarity, Angles & Triangles**

(a) $\angle LAD = \angle LCB$ (angles in same segment)
 $\angle LDA = \angle LBC$ (angles in same segment)
 $\angle ALD = \angle CLB = 90^\circ$ (vertically opposite angles)
 ∴ $\triangle LAD$ and $\triangle LCB$ are similar (AAA) (Shown)

(b) (i) $\angle CNO = 90^\circ$ ($ON \perp BC$ ∵ N is midpt of BC of isosceles $\triangle OBC$)

(ii) $\angle DCB = \angle DAB = 58^\circ$ (angles in same segment)

$\angle CON = 180^\circ - \angle CNO - (\angle DCO + \angle DCB)$ (sum of \angle s in a triangle)
 $= 180^\circ - 90^\circ - (18^\circ + 58^\circ)$
 $= 14^\circ$

$\angle CNO = 90^\circ$ from (b)(i)

(iii) $\angle CBA = 180^\circ - \angle CLB - \angle DCB$ (sum of \angle s in a triangle)
 $= 180^\circ - 90^\circ - 58^\circ$
 $= 32^\circ$

$\angle DCB = 58^\circ$ from (b)(ii)

(iv) $\angle CDO = \angle DCO = 18^\circ$ (base \angle s of isosceles $\triangle DCO$)
 $\angle ADC = \angle CBA$ (angles in same segment)
 $= 32^\circ$

$\angle CBA = 32^\circ$ from (b)(iii)

$\angle ADO = \angle ADC - \angle CDO$
 $= 32^\circ - 18^\circ$
 $= 14^\circ$

7. Topics: Geometrical Properties of Circles, Trigonometry

- (a) (i) $\tan \angle AOC = \frac{AC}{OC}$ ($OC \perp AC \therefore OC$ is perpendicular bisector of chord AB)

$$\begin{aligned} &= \frac{40}{50} \\ \angle AOC &= 0.6747 \text{ rad} \\ \therefore \angle AOB &= 2 \times \angle AOC \\ &= 2 \times 0.6747 \text{ rad} \\ &\approx 1.349 \text{ rad} \\ &\approx \mathbf{1.35 \text{ rad (3 sig. fig.)}} \end{aligned}$$

- (ii) Using Pythagoras' theorem for $\triangle OAC$,

$$\begin{aligned} AO \text{ (length of radius of sector } OAB) &= \sqrt{40^2 + 50^2} \text{ cm} \\ &= \sqrt{4100} \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{Area of window} &= \text{Area of sector } OAB - \text{Area of } \triangle OAB \\ &= \frac{1}{2} \times AO^2 \times \angle AOB - \frac{1}{2} \times AB \times OC \\ &= \frac{1}{2} (\sqrt{4100})^2 (1.349) - \frac{1}{2} (80) (50) \\ &\approx \mathbf{766 \text{ cm}^2 \text{ (3 sig. fig.)}} \end{aligned}$$

- (b) (i) Using cosine rule for $\triangle DEX$,

$$\begin{aligned} EX^2 &= DE^2 + DX^2 - 2(DE)(DX) \cos \angle EDX \\ &= 80^2 + 80^2 - 2(80)(80) \cos 38^\circ \\ \therefore EX &\approx 52.09 \text{ cm} \\ &\approx \mathbf{52.1 \text{ cm (3 sig. fig.)}} \end{aligned}$$

- (ii) Using Pythagoras' theorem,

$$\begin{aligned} DF = DY &= \sqrt{200^2 + 80^2} \\ &= \sqrt{46400} \text{ cm} \end{aligned}$$

Using cosine rule for $\triangle FDY$,

$$\begin{aligned} FY^2 &= DF^2 + DY^2 - 2(DF)(DY) \cos \angle FDY \\ \cos \angle FDY &= \frac{(\sqrt{46400})^2 + (\sqrt{46400})^2 - (52.09)^2}{2 \sqrt{46400} \sqrt{46400}} \\ &= 0.97076 \\ \therefore \angle FDY &\approx \mathbf{13.9^\circ \text{ (3 sig. fig.)}} \end{aligned}$$

$FY = EX \approx 52.09 \text{ cm}$
from (b)(i)

8. Topic: Mensuration

- (a) (i) Using Pythagoras' theorem,

$$\begin{aligned} \text{Slant height, } s &= \sqrt{\left(\frac{0.8}{2}\right)^2 + 2^2} \text{ cm} \\ &= \sqrt{0.4^2 + 2^2} \text{ cm} \\ &\approx 2.0396 \text{ cm} \\ &\approx \mathbf{2.04 \text{ cm (3 sig. fig.)}} \end{aligned}$$

Radius of pencil = $\frac{0.8}{2}$ cm

- (ii) Total surface area

$$\begin{aligned} &= \text{Area of cone} + \text{area of cylinder} + \text{area of circular base} \\ &= \pi(0.4)(2.0396) + 2\pi(0.4)(16) + \pi(0.4)^2 \\ &\approx \mathbf{43.3 \text{ cm}^2 \text{ (3 sig. fig.)}} \end{aligned}$$

Sub $s = 2.0396$
from (a)(i) into
area of cone = πrs .

- (b) Volume of pencil

$$\begin{aligned} &= \text{Volume of cone} + \text{volume of cylinder} \\ &= \frac{1}{3} \pi (0.4)^2 (2) + \pi (0.4)^2 (16) \\ &\approx 8.378 \text{ cm}^3 \\ &\approx \mathbf{8.38 \text{ cm}^3 \text{ (3 sig. fig.)}} \end{aligned}$$

- (c) (i) Width of box = $6 \times$ pencil diameter = $6 \times 0.8 \text{ cm} = 4.8 \text{ cm}$
Height of box = $2 \times$ pencil diameter = $2 \times 0.8 \text{ cm} = 1.6 \text{ cm}$
Length of box = $1 \times$ pencil length = $(16+2) \text{ cm} = 18 \text{ cm}$
 \therefore volume of box = $4.8 \text{ cm} \times 1.6 \text{ cm} \times 18 \text{ cm}$
 $= \mathbf{138.24 \text{ cm}^3 \text{ (Shown)}}$

- (ii) Volume of box not occupied by the pencils

$$\begin{aligned} &= \text{Volume of box} - \text{total volume of 12 pencils in box} \\ &= 138.24 \text{ cm}^3 - 12 \times 8.378 \text{ cm}^3 \\ &= 37.704 \text{ cm}^3 \end{aligned}$$

Volume of each pencil
 $\approx 8.378 \text{ cm}^3$ from (b)

\therefore % of the volume not occupied by the pencils

$$\begin{aligned} &= \frac{37.704}{138.24} \times 100\% \\ &\approx \mathbf{27.3 \%} \end{aligned}$$

9. **Topic: Graphical Solution of Equations**

- (a) Sub $x = 4$ into $y = \frac{1}{5}x(12 - x^2)$: $p = \frac{1}{5}(4)(12 - 4^2) = -3.2$
 (b) See graph.
 (c) Plot $y = 1$ for the range $-3 \leq x \leq 4$.

From graph,

$y = \frac{1}{5}x(12 - x^2)$ intersects $y = 1$ at $x = 0.42, 3.23$

\therefore Solution of $\frac{1}{5}x(12 - x^2) = 1$: $x = 0.42, 3.23$

Check: $x^3 - 12x + 5 = 0$
 $\Rightarrow x = 3.23, -3.66, 0.42$

- (d) From graph, gradient of tangent at $(3, 1.8) = \frac{4 - (-0.5)}{2.25 - 3.75} = -3$

AMaths students:

Check: $\frac{dy}{dx} = \frac{12}{5} - \frac{3x^2}{5}$
 Sub $x = 3 \Rightarrow \frac{dy}{dx} = -3$

- (e) Since $2x + y = 2$ is linear, sub the values of $x = -1$ and $x = 3$ to obtain the y -values of the two points:

x	-1	3
y	4	-4

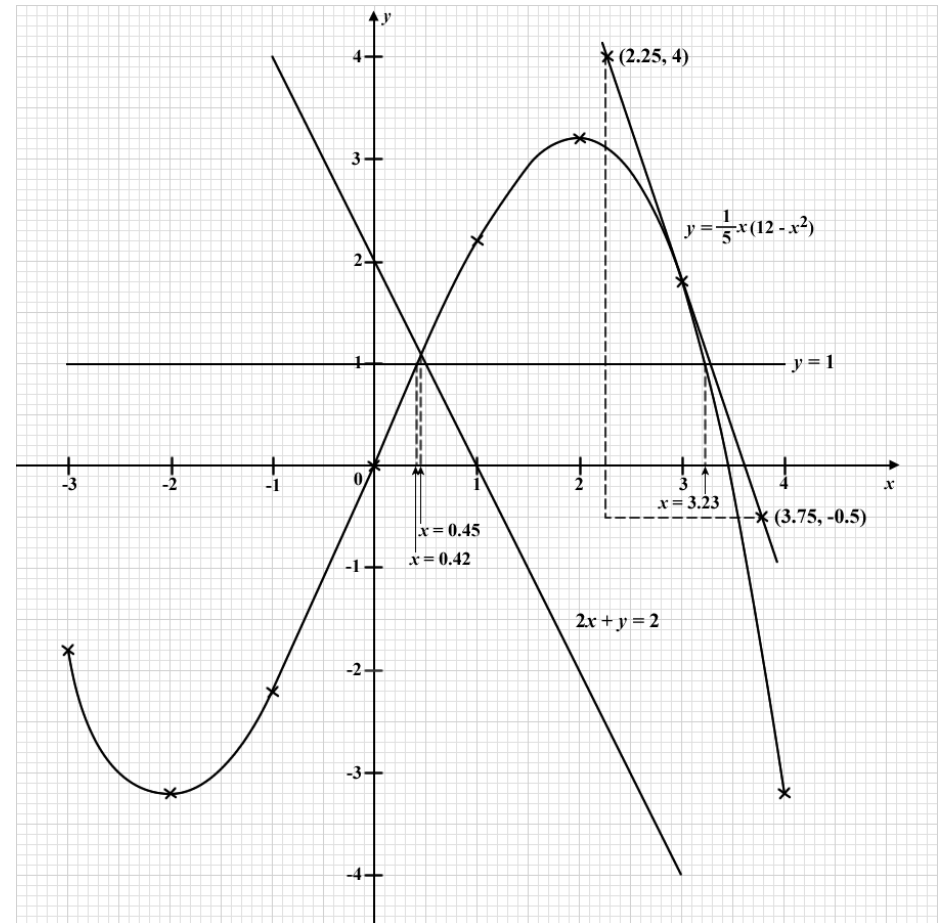
Join up these two points to get the graph of $2x + y = 2$.

- (f) (i) From graph, x -coordinate = **0.45**

(ii) $\frac{1}{5}x(12 - x^2) = 2 - 2x$
 $x^3 - 22x + 10 = 0 \dots\dots\dots (1)$

Comparing coefficients of (1) with $x^3 + Ax^2 + Bx + C = 0$,

$A = 0, B = -22, C = 10$



10. Topics: Data Analysis (Statistics, Cumulative Frequency Distribution), Probability

(a) (i) From the cumulative frequency graph,

Mass (x kg)	$4 \leq x < 8$	$8 \leq x < 12$	$12 \leq x < 16$	$16 \leq x < 20$	$20 \leq x < 24$
Frequency	3	7	14	11	5

freq. ($x < 8$) –
freq. ($x \leq 4$)
= $3 - 0$

freq. ($x < 12$) –
freq. ($x \leq 8$)
= $10 - 3$

freq. ($x < 16$) –
freq. ($x \leq 12$)
= $24 - (7+3)$

freq. ($x < 24$) –
freq. ($x \leq 20$)
= $40 - (7+3+14+11)$

freq. ($x < 20$) –
freq. ($x \leq 16$)
= $35 - (7+3+14)$

(ii) (a) Mean mass = $\frac{\sum fx}{\sum f}$

$$= \frac{3(6)+7(10)+14(14)+11(18)+5(22)}{40}$$

$$= 14.8$$

Use the mid-value of each interval for x .

(b) Standard deviation = $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

$$= \sqrt{\frac{3(6)^2+7(10)^2+14(14)^2+11(18)^2+5(22)^2}{40} - 14.8^2}$$

$$= 4.4$$

$\frac{\sum fx}{\sum f} = 14.8$ from (ii)(a)

(iii) The 2nd curve will have an overall gentler slope (due to its larger standard deviation), lying above the original curve for $x < 15$ and below the original curve for $x > 15$, and intersecting the original curve at $x = 15$ (since they have the same median).

(b) (i)

	1	2	3	4	5	6
1		1,2	1,3 [♦]	1,4	1,5	1,6 [♦]
2	2,1		2,3 [♦]	2,4 [♠]	2,5	2,6 ^{♠♠}
3	3,1 [♦]	3,2 [♦]		3,4 [♦]	3,5 ^{♠♦}	3,6 [♦]
4	4,1	4,2 [♠]	4,3 [♦]		4,5	4,6 ^{♠♦}
5	5,1	5,2	5,3 ^{♠♦}	5,4		5,6 [♦]
6	6,1 [♦]	6,2 ^{♠♠♦}	6,3 [♦]	6,4 ^{♠♦}	6,5 [♦]	

♠ - both balls even

♣ - sum is 8

♦ - at least one is multiple of 3

Without replacement \Rightarrow
(1, 1), (2, 2) ... (6, 6)
outcomes are impossible.

From the possibility diagram, total no. of possible outcomes = 30

(ii) (a) P (both have even number) = $\frac{6}{30}$

$$= \frac{1}{5}$$

$\frac{\text{\# of } \spadesuit}{\text{Total \# of outcomes}}$
Check: $\frac{1}{6} \times \frac{1}{5} \times 6$

(b) P (sum of numbers drawn is 8) = $\frac{4}{30}$

$$= \frac{2}{15}$$

$\frac{\text{\# of } \clubsuit}{\text{Total \# of outcomes}}$
Check: $\frac{1}{6} \times \frac{1}{5} \times 4$

(c) P (product is 7) = $\frac{0}{30} = 0$

(d) P (at least one of the no. drawn is a multiple of 3) = $\frac{8}{30}$

$$= \frac{4}{15}$$

$\frac{\text{\# of } \blacklozenge}{\text{Total \# of outcomes}}$
Check: $\frac{1}{6} \times \frac{1}{5} \times 18$