

ADDITIONAL MATHEMATICS
Paper 2 Suggested Solutions

4038/02
October/November 2008

1. **Topic: Exponential Functions**

(i) Given: $V = 10000e^{-pt}$

When the man bought the motorcycle, $t = 0$.

$$\therefore \text{value of motorcycle when bought} = 10000e^0 = \mathbf{\$10000}$$

(ii) When $t = 12$, $v = 4000$:

$$4000 = 10000e^{-p(12)}$$

$$0.4 = e^{-12p}$$

$$\ln 0.4 = -12p \ln e$$

$$p = \frac{\ln 0.4}{-12}$$

$$\approx 0.076358$$

$$V = 10000e^{-0.076358t} \dots\dots\dots (1)$$

$$\text{Sub } t = 18 \text{ into (1): Value after 18 months} = 10000e^{-0.076358(18)}$$

$$\approx 2529.8$$

$$\approx \mathbf{\$2530 (3 \text{ s.f.})}$$

(iii) Sub $v = 1000$ into (1):

$$1000 = 10000e^{-0.076358t}$$

$$0.1 = e^{-0.076358t}$$

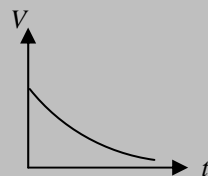
$$\ln 0.1 = -0.076358t \ln e$$

$$t = \frac{\ln 0.1}{-0.076358}$$

$$\approx 30.2 \text{ months}$$

$$\approx 30 \text{ months (nearest month)}$$

Note: $V < \$1000$ when $t = 31$ months, since the e^{-pt} curve describes the depreciation of value over time.



\therefore age of motorcycle when expected value is \$1000 is 30 months.

2. **Topic: Quadratic Equations (Sum & Product of Roots)**

$$2x^2 - 4x + 3 = 0 \Rightarrow a = 2, b = -4, c = 3$$

$$\text{Sum of roots: } \alpha + \beta = -\frac{b}{a} = -\frac{-4}{2} = 2$$

$$\text{Product of roots: } \alpha\beta = \frac{c}{a} = \frac{3}{2}$$

New sum of roots:

$$(\alpha^2 + 2) + (\beta^2 + 2) = \alpha^2 + \beta^2 + 4$$

$$= [(\alpha + \beta)^2 - 2\alpha\beta] + 4$$

$$= (2)^2 - 2\left(\frac{3}{2}\right) + 4$$

$$= 5$$

Sub $\alpha + \beta = 2, \alpha\beta = \frac{3}{2}$

New product of roots:

$$(\alpha^2 + 2)(\beta^2 + 2) = \alpha^2\beta^2 + 2\alpha^2 + 2\beta^2 + 4$$

$$= (\alpha\beta)^2 + 2(\alpha^2 + \beta^2) + 4$$

$$= (\alpha\beta)^2 + 2[(\alpha + \beta)^2 - 2\alpha\beta] + 4$$

$$= \frac{9}{4} + 2\left[4 - 2\left(\frac{3}{2}\right)\right] + 4$$

$$= \frac{9}{4} + 2 + 4$$

$$= 8\frac{1}{4} = \frac{33}{4}$$

Useful expression:
 $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

Sub $\alpha + \beta = 2, \alpha\beta = \frac{3}{2}$

\therefore equation with roots $\alpha^2 + 2, \beta^2 + 2$:

$$x^2 - 5x + \frac{33}{4} = 0$$

$$\Rightarrow 4x^2 - 20x + 33 = 0$$

$x^2 - (\text{Sum of roots})x + (\text{Product of roots}) = 0$

3. Topic: Trigonometry (Trigonometric Identities & Equations)

(i) $\tan A + \cot A = 2 \operatorname{cosec} 2A$

$$\begin{aligned} \text{L.H.S.: } \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} &= \frac{\sin^2 A + \cos^2 A}{\cos A \sin A} \\ &= \frac{1}{\sin A \cos A} \\ &= \frac{1}{\frac{2 \sin A \cos A}{2}} \\ &= \frac{2}{\sin 2A} \\ &= 2 \operatorname{cosec} 2A \\ &= \text{R.H.S. (Proved)} \end{aligned}$$

(ii) $\tan A + \cot A = 3$

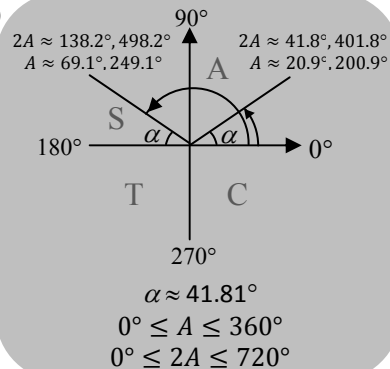
From (i): $2 \operatorname{cosec} 2A = 3$

$$\begin{aligned} \frac{2}{\sin 2A} &= 3 \\ \sin 2A &= \frac{2}{3} \end{aligned}$$

$$\Rightarrow \text{Basic angle } \alpha = \sin^{-1}\left(\frac{2}{3}\right) \approx 41.81^\circ$$

$$\begin{aligned} 2A &\approx 41.81^\circ, 180^\circ - 41.81^\circ, \\ &\quad 360^\circ + 41.81^\circ, 360^\circ + 180^\circ - 41.81^\circ \\ &\approx 41.81^\circ, 138.19^\circ, 401.81^\circ, 498.19^\circ \end{aligned}$$

$$\therefore A = 20.9^\circ, 69.1^\circ, 200.9^\circ, 249.1^\circ$$



4. Topic: Logarithms

(i) $2 + \log_3(3x - 7) = \log_3(2x - 3)$

$$\log_3(2x - 3) - \log_3(3x - 7) = 2$$

$$\log_3\left(\frac{2x-3}{3x-7}\right) = 2$$

$$\frac{2x-3}{3x-7} = 3^2$$

$$2x - 3 = 9(3x - 7)$$

$$2x - 3 = 27x - 63$$

$$60 = 25x$$

$$x = 2.4$$

Quotient Law:

$$\log_a m - \log_a n = \log_a \frac{m}{n}$$

$$y = a^x \Leftrightarrow x = \log_a y$$

(ii) $3 \log_5 y - \log_5 5 = 2$

$$3 \log_5 y - \frac{1}{\log_5 y} = 2$$

Let $\log_5 y = x$.

$$3x - \frac{1}{x} = 2$$

$$3x^2 - 1 = 2x$$

$$(3x + 1)(x - 1) = 0$$

$$x = -\frac{1}{3} \text{ or } 1$$

$$\Rightarrow \log_5 y = -\frac{1}{3} \text{ or } 1$$

$$y = 5^{-\frac{1}{3}} \text{ or } 5^1$$

$$\therefore y = \frac{1}{\sqrt[3]{5}} \text{ or } 5$$

Change of Base of Log:

$$\log_a b = \frac{1}{\log_b a}$$

5. Topic: Polynomials (Factor Theorem & Remainder Theorem)

(i) $f(x) = 2(x^2 - 3x + 1)(x + 1)(x - 2)$
 $= (2x^2 - 6x + 2)(x^2 - x - 2)$
 $= 2x^4 - 2x^3 - 4x^2 - 6x^3 + 6x^2 + 12x + 2x^2 - 2x - 4$
 $= 2x^4 - 8x^3 + 4x^2 + 10x - 4$

Factor Theorem:
 $f(a) = 0 \Leftrightarrow (x - a)$ is a factor of $f(x)$

(ii) $f(x) = 0$

Check: $f(-1) = 2(-1)^4 - 8(-1)^3 + 4(-1)^2 + 10(-1) - 4 = 0$

$2(x^2 - 3x + 1)(x + 1)(x - 2) = 0$

$x = \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2}$ or -1 or 2
 $= \frac{3 \pm \sqrt{5}}{2}, -1, 2$

\therefore No. of real roots = 4

(iii) $f(x) = 2(x^2 - 3x + 1)(x + 1)(x - 2)$

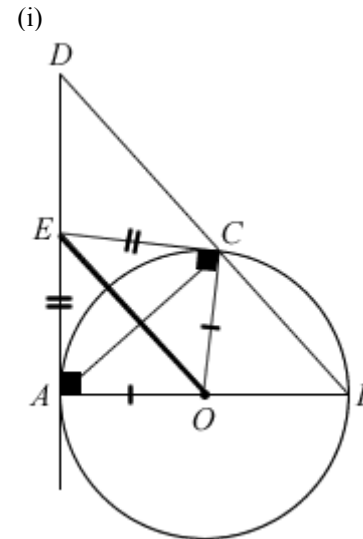
When $x = \frac{1}{2}$,

Remainder Theorem:
 $f(x)$ divided by $(x - a) \Rightarrow$ remainder is $f(a)$

$f\left(\frac{1}{2}\right) = 2\left[\frac{1}{4} - \frac{3}{2} + 1\right]\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)$
 $= 2\left(-\frac{1}{4}\right)\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)$
 $= 1\frac{1}{8}$

Remainder = $1\frac{1}{8}$

6. Topic: Geometric Proofs

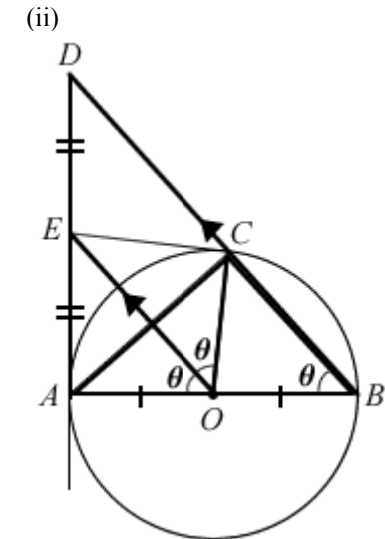


$AE = CE$ (Tangents from a common external point E)

$AO = CO$ (Radius of circle)

$\angle OAE = \angle OCE$
 $= 90^\circ$ (Radius \perp Tangent)

$\therefore \triangle AEO \cong \triangle CEO$ (SAS)



Let $\angle AOE = \theta$.

$\angle COE = \theta$ ($\triangle AEO \cong \triangle CEO$)

$\angle AOC = 2\theta$

$\Rightarrow \angle ABC = \theta$

(\angle at centre =

$2 \times \angle$ at circumference)

$\Rightarrow EO \parallel DB$ ($\angle AOE = \angle ABC = \theta$
 - corresponding \angle s)

\therefore By Midpoint Theorem,
 E is the mid-pt of AD .

7. **Topic: Trigonometry (Trigonometric Functions)**

(i) Amplitude = 4

(ii) Period = $\frac{360^\circ}{2 \text{ cycles}} = 180^\circ$

(iii) Minimum point occurs when $\cos 2x = -1$

$\Rightarrow 2x = \cos^{-1}(-1)$

$2x = 180^\circ$

$x = 90^\circ$

$\Rightarrow y = 4(-1) - 2 = -6$

\therefore coordinates of the minimum point of the curve is $(90^\circ, -6)$

(iv) When $y = 0$, $4\cos 2x - 2 = 0$

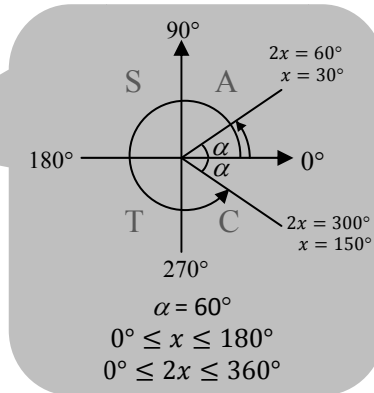
$4\cos 2x = 2$

$\cos 2x = \frac{1}{2}$

Basic angle $\alpha = 60^\circ$

$2x = 60^\circ, 300^\circ$

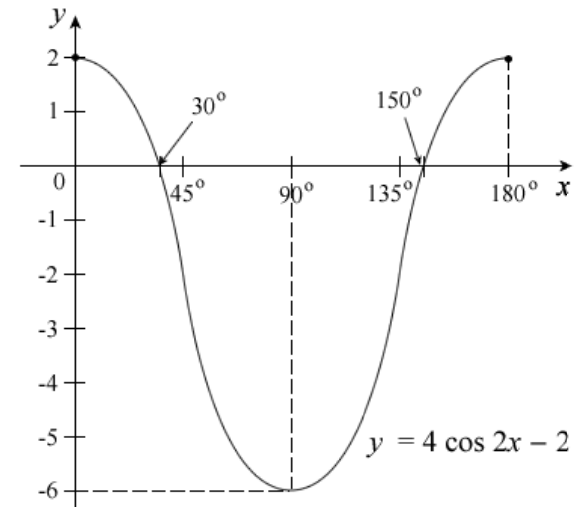
$x = 30^\circ, 150^\circ$



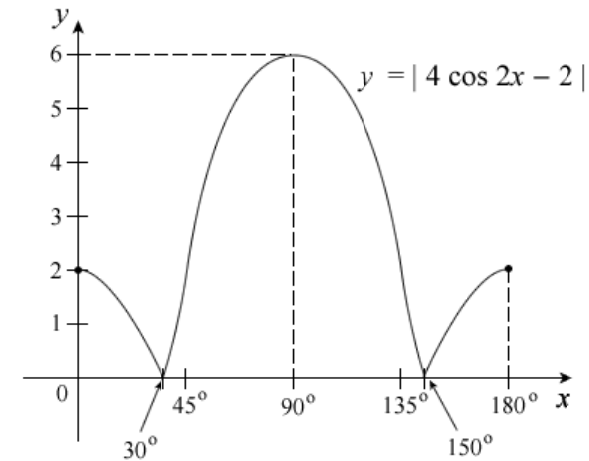
\therefore coordinates where curve meets the x -axis are $(30^\circ, 0)$ and $(150^\circ, 0)$

$y = a \cos bx + c$: Amplitude = a
Period = $\frac{360^\circ}{b}$

(v)



(vi)



8. Topic: Applications of Differentiation & Integration (Maxima & Minima, Area Under Curve)

(i) $y = x^3 - ax + b$

$$\frac{dy}{dx} = 3x^2 - a$$

From the diagram, minimum point is (2, 0)

 \Rightarrow Sub $\frac{dy}{dx} = 0$ and $x = 2$ into $\frac{dy}{dx}$:

$$3(2)^2 - a = 0$$

$$12 - a = 0$$

$$a = 12$$

Sub (2, 0) into y : $0 = 2^3 - 12(2) + b$

$$0 = 8 - 24 + b$$

$$b = 16$$

 $\therefore a = 12, b = 16$

(ii) From (i),

$$y = x^3 - 12x + 16$$

$$\frac{dy}{dx} = 3x^2 - 12$$

At maximum point,

$$\frac{dy}{dx} = 3x^2 - 12 = 0$$

$$3(x^2 - 4) = 0$$

$$3(x + 2)(x - 2) = 0$$

 $x = -2$ or 2 (rejected \because max. point occurs when $x < 0$ in the diagram)Sub $x = -2$ into y :

$$y = (-2)^3 - 12(-2) + 16$$

$$= -8 + 24 + 16$$

$$= 32$$

 \therefore coordinates of maximum point is $(-2, 32)$.Check: $\frac{d^2y}{dx^2} = 6x$ Sub $x = -2$ into $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = 6(-2) = -12 < 0$$

 $\Rightarrow (-2, 32)$ is a maximum point(iii) Area of shaded region $= \int_0^2 (x^3 - 12x + 16) dx$

$$= \left[\frac{x^4}{4} - \frac{12x^2}{2} + 16x \right]_0^2$$

$$= \left[\frac{x^4}{4} - 6x^2 + 16x \right]_0^2$$

$$= \left[\frac{(2)^4}{4} - 6(2)^2 + 16(2) \right] - \left[\frac{(0)^4}{4} - 6(0)^2 + 16(0) \right]$$

$$= 4 - 24 + 32 - 0$$

$$= 12 \text{ units}^2$$

9. Topic: Further Trigonometric Identities (R-Formula)

(i) From the diagram,

$$\angle OAD = \theta \text{ (corresponding } \angle \text{s)}$$

$$\sin \theta = \frac{OD}{4}$$

$$OD = 4 \sin \theta$$

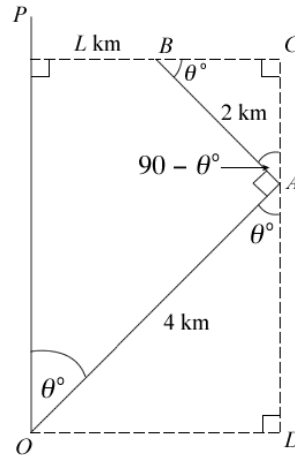
$$\begin{aligned} \angle ABC &= 180^\circ - 90^\circ - \angle BAC \\ &= 180^\circ - 90^\circ - (180^\circ - 90^\circ - \theta) \\ &= \theta \end{aligned}$$

$$\cos \theta = \frac{BC}{2}$$

$$BC = 2 \cos \theta$$

$$\therefore L = OD - BC$$

$$= 4 \sin \theta - 2 \cos \theta \text{ (Shown)}$$



$$\begin{aligned} \text{(ii) } 4 \sin \theta - 2 \cos \theta &= R \sin (\theta - \alpha) \\ &= R (\sin \theta \cos \alpha - \cos \theta \sin \alpha) \\ &= R \cos \alpha \sin \theta - R \sin \alpha \cos \theta \end{aligned}$$

Comparing coefficients,

$$4 = R \cos \alpha \dots\dots\dots (1)$$

$$2 = R \sin \alpha \dots\dots\dots (2)$$

$$\frac{(2)}{(1)}: \frac{\sin \alpha}{\cos \alpha} = \frac{2}{4}$$

$$\tan \alpha = \frac{1}{2}$$

$$\alpha \approx 26.6^\circ$$

$$(1)^2 + (2)^2: R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 4^2 + 2^2$$

$$R^2 (\cos^2 \alpha + \sin^2 \alpha) = 16 + 4$$

$$R^2 = 20$$

$$R = \sqrt{20} \text{ or } -\sqrt{20} \text{ (rejected)}$$

$$\therefore L = \sqrt{20} \sin (\theta - 26.6^\circ)$$

Addition Formula:
 $\sin (A - B) = \sin A \cos B - \cos A \sin B$

(iii) When $L = 3$,

$$\sqrt{20} \sin (\theta - 26.6^\circ) = 3$$

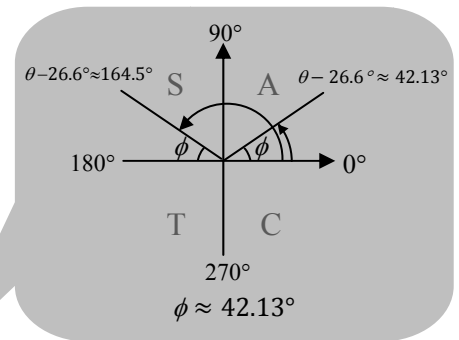
$$\sin (\theta - 26.6^\circ) = \frac{3}{\sqrt{20}}$$

$$\text{Basic angle } \phi \approx 42.13^\circ$$

$$\theta - 26.6^\circ \approx 42.13^\circ, 180^\circ - 42.13^\circ$$

$$\theta \approx 68.7^\circ, 164.5^\circ \text{ (rejected } \because \theta \text{ is acute)}$$

$$\therefore \theta \approx 68.7^\circ \text{ (3 sig. fig.)}$$



10. Topics: Coordinate Geometry, Integration,
Applications of Differentiation (Rate of Change)

(i) $\frac{dy}{dx} = \frac{6}{(2x-1)^2}$

Sub $x = 2$ into $\frac{dy}{dx}$:

Gradient of curve at $P(2, 9) = \frac{6}{3^2} = \frac{2}{3}$

Gradients m_1 and m_2 of two \perp lines $\Leftrightarrow m_1 m_2 = -1$

\Rightarrow Gradient of normal $= -\frac{3}{2}$

Equation of normal to the curve at P :

$y - 9 = -\frac{3}{2}(x - 2)$

Equation of line with gradient m and point (x_1, y_1) :
 $(y - y_1) = m(x - x_1)$

$y = -\frac{3}{2}x + 3 + 9$

$y = -\frac{3}{2}x + 12 \dots\dots\dots (1)$

At $Q(0, y)$,

Sub $x = 0$ into (1): $y = -\frac{3}{2}(0) + 12$

$= 12$

$\Rightarrow Q = (0, 12)$

At $R(x, 0)$,

Sub $y = 0$ into (1): $-\frac{3}{2}x + 12 = 0$

$\frac{3}{2}x = 12$

$x = 8$

$\Rightarrow R = (8, 0)$

Midpoint of (x_1, y_1) and (x_2, y_2) :

\therefore Mid-point of $QR = \left(\frac{0+8}{2}, \frac{12+0}{2}\right)$

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= (4, 6)$

(ii) $y = \int \frac{6}{(2x-1)^2} dx$

$= \int 6(2x - 1)^{-2} dx$

$= \frac{6(2x-1)^{-1}}{2(-1)} + c$

$= \frac{3(2x-1)^{-1}}{-1} + c$

$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$

$y = -\frac{3}{2x-1} \dots\dots\dots (2)$

Since $P(2, 9)$ lies on the curve, sub $x = 2, y = 9$ into (2):

$9 = -\frac{3}{2(2)-1} + c$

$9 = -1 + c$

$c = 10$

$\therefore y = -\frac{3}{2x-1} + 10$

(iii) x -coordinate increases at 0.03 units per second

$\Rightarrow \frac{dx}{dt} = 0.03$

Rate of change of y -coordinate:

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Chain Rule:

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{6}{(2x-1)^2} \times 0.03$

At $P(2, 9)$, sub $x = 2$ into $\frac{dy}{dt}$:

$\frac{dy}{dt} = \frac{6}{(2(2)-1)^2} \times 0.03$

$= 0.02$ units per second

$\therefore y$ -coordinate increases at 0.02 units per second.

11. Topics: Coordinate Geometry, Circles

(i) Centre of C_1 , $O = (0, 0)$

Radius of C_1 , $OP = \sqrt{(0-8)^2 + (0+6)^2}$
= 10 units

\therefore Equation of C_1 : $(x-0)^2 + (y-0)^2 = 10^2$
 $x^2 + y^2 = 100$

Length of Line Segment
 $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Equation of circle with
centre (a, b) and radius r :
 $(x-a)^2 + (y-b)^2 = r^2$

(ii) Center of C_2 , $Q =$ Midpoint of OP

$= \left(\frac{0+8}{2}, \frac{0+(-6)}{2}\right)$
 $= (4, -3)$

Radius of C_2 , $QP = \sqrt{(4-8)^2 + (-3+6)^2}$
= 5 units

\therefore Equation of C_2 : $(x-4)^2 + (y+3)^2 = 5^2$
 $x^2 - 8x + 16 + y^2 + 6y + 9 = 25$
 $x^2 + y^2 - 8x + 6y = 0$

Midpoint of (x_1, y_1) and
 (x_2, y_2) :

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

(iii) Gradient of $OP = \frac{0+6}{0-8} = \frac{3}{-4}$

\Rightarrow Gradient of $AB = \frac{4}{3}$

Using $Q(4, -3)$ from (ii):

Equation of AB : $y + 3 = \frac{4}{3}(x - 4)$
 $y = \frac{4}{3}x - \frac{25}{3}$ (1)

Gradients m_1 and m_2 of two
 \perp lines $\Leftrightarrow m_1 m_2 = -1$

Equation of line with
gradient m and point (x_1, y_1) :
 $(y - y_1) = m(x - x_1)$

A and B also lie on C_1 with equation obtained in (i):

$x^2 + y^2 = 100$ (2)

Sub (1) into (2):

$x^2 + \left(\frac{4}{3}x - \frac{25}{3}\right)^2 = 100$
 $x^2 + \frac{16x^2}{9} - \frac{200}{9}x + \frac{625}{9} = 100$
 $\frac{25}{9}x^2 - \frac{200}{9}x + \frac{625}{9} = 0$
 $25x^2 - 200x - 275 = 0$
 $x^2 - 8x - 11 = 0$

$x = \frac{8 \pm \sqrt{64 - 4(1)(-11)}}{2(1)}$
 $= \frac{8 \pm \sqrt{108}}{2}$
 $= \frac{8 \pm 6\sqrt{3}}{2}$
 $= 4 \pm 3\sqrt{3}$

\therefore x -coordinates of A and B are $4 + 3\sqrt{3}$ and $4 - 3\sqrt{3}$ respectively.