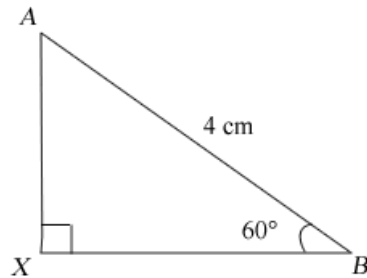


ADDITIONAL MATHEMATICS
Paper 1 Suggested Solutions

4038/01
October/November 2008

1. **Topic: Trigonometry (Trigonometric Ratios)**

(i) $\angle ABX = 60^\circ$
 $\sin \angle ABX = \frac{AX}{AB}$
 $\sin 60^\circ = \frac{AX}{4}$
 $AX = \frac{\sqrt{3}}{2} \times 4$
 $= 2\sqrt{3} \text{ cm}$



(ii) Using Pythagoras' theorem in $\triangle ABX$,

$$AB^2 = AX^2 + BX^2$$

$$16 = (2\sqrt{3})^2 + BX^2$$

$$BX^2 = 16 - 12$$

$$= 4$$

$$BX = 2 \text{ cm}$$

$AX = 2\sqrt{3}$ from (i)
 $CX = BC + BX$
 $= 2 + 2 = 4 \text{ cm}$

In $\triangle AXC$, $\tan \angle ACB = \frac{AX}{CX}$
 $= \frac{2\sqrt{3}}{4}$
 $\angle ACB = \tan^{-1} \left(\frac{\sqrt{3}}{2} \right)$ (Shown)

2. **Topics: Indices, Simultaneous Equations**

$$9^x(27)^y = 1$$

$$(3^{2x})(3^{3y}) = 3^0$$

$$3^{2x+3y} = 3^0$$

Comparing the indices: $2x + 3y = 0$
 $6y = -4x \dots\dots\dots (1)$

$$8^y \div (\sqrt{2})^x = 16\sqrt{2}$$

$$2^{3y} \div 2^{\frac{1}{2}x} = (2^4) \left(2^{\frac{1}{2}} \right)$$

$$2^{3y-\frac{1}{2}x} = 2^{4+\frac{1}{2}}$$

Comparing the indices: $3y - \frac{1}{2}x = 4 + \frac{1}{2}$
 $3y - \frac{1}{2}x = \frac{9}{2}$
 $6y - x = 9 \dots\dots\dots (2)$

Sub (1) into (2): $-4x - x = 9$
 $x = -\frac{9}{5}$

Sub $x = -\frac{9}{5}$ into (1): $6y = -4 \left(-\frac{9}{5} \right)$
 $= \frac{36}{5}$
 $y = \frac{6}{5}$

$\therefore x = -\frac{9}{5}, y = \frac{6}{5}$

3. Topic: Simultaneous Equations (Solution by Inverse Matrix Method)

$$A = \begin{pmatrix} 7 & -8 \\ 1 & 6 \end{pmatrix}$$

$$A^{-1} = \frac{1}{(7)(6) - (-8)(1)} \begin{pmatrix} 6 & 8 \\ -1 & 7 \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 6 & 8 \\ -1 & 7 \end{pmatrix}$$

Given: $7q - 8p = 11$
 $q + 6p = -7$

Expressing the above as a matrix equation,

$$\begin{pmatrix} 7 & -8 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix} \quad A \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 11 \\ -7 \end{pmatrix}$$

Multiply both sides by A^{-1} ,

$$\begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 6 & 8 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 11 \\ -7 \end{pmatrix}$$

$$= \frac{1}{50} \begin{pmatrix} 10 \\ -60 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$

$$\therefore q = \frac{1}{5}, p = -\frac{6}{5}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1}A \begin{pmatrix} q \\ p \end{pmatrix} = A^{-1} \begin{pmatrix} 11 \\ -7 \end{pmatrix}$$

$$I \begin{pmatrix} q \\ p \end{pmatrix} = A^{-1} \begin{pmatrix} 11 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} q \\ p \end{pmatrix} = A^{-1} \begin{pmatrix} 11 \\ -7 \end{pmatrix}$$

4. Topic: Differentiation & Integration

$$(i) \frac{d}{dx}(x^3 \ln x) = x^3 \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x^3)$$

$$= x^3 \times \frac{1}{x} + 3x^2 \ln x$$

$$= x^2 + 3x^2 \ln x$$

Product Rule:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$(ii) \int x^2 \ln x \, dx = \frac{1}{3} \int 3x^2 \ln x \, dx$$

$$= \frac{1}{3} \left[\int (3x^2 \ln x + x^2 - x^2) \, dx \right]$$

$$= \frac{1}{3} \left[\int (3x^2 \ln x + x^2) \, dx - \int x^2 \, dx \right]$$

$$= \frac{1}{3} \int (3x^2 \ln x + x^2) \, dx - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + c$$

From (i): $\frac{d}{dx}(x^3 \ln x) = x^3 + 3x^2 \ln x \Rightarrow$
 $\int (3x^2 \ln x + x^2) \, dx = x^3 \ln x + c$

5. Topics: Partial Fractions, Applications of Differentiation (Gradients)

$$(i) \frac{8x-46}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$

Multiply both sides by $(x-5)(x+1)$:

$$8x - 46 = A(x+1) + B(x-5)$$

Let $x = -1$: $-8 - 46 = B(-6)$
 $B = 9$

Let $x = 5$: $40 - 46 = A(6)$
 $A = -1$

$$\therefore \frac{8x-46}{(x-5)(x+1)} = -\frac{1}{x-5} + \frac{9}{x+1}$$

$$(ii) \text{ From (i), } y = \frac{8x-46}{(x-5)(x+1)}$$

$$= -\frac{1}{x-5} + \frac{9}{x+1}$$

$$\frac{dy}{dx} = (x-5)^{-2} - 9(x+1)^{-2}$$

When $x = 2$, $\frac{dy}{dx} = (2-5)^{-2} - 9(2+1)^{-2} = \frac{1}{9} - \frac{9}{9}$

$$= -\frac{8}{9}$$

6. **Topic: Applications of Differentiation & Integration (Kinematics)**

(i) $v = 6t - \frac{1}{2}t^2$

Since the cyclist is at rest, $v = 0$ at point B.

$$\Rightarrow 6t - \frac{1}{2}t^2 = 0$$

$$\frac{1}{2}t(12 - t) = 0$$

$$t = 0 \text{ (reject) or } 12$$

\therefore Time taken from A to B = **12 s**

(ii) Distance $AB = \int_0^{12} (6t - \frac{1}{2}t^2) dt$

$$= \left[3t^2 - \frac{1}{6}t^3 \right]_0^{12}$$

$$= 3(12)^2 - \frac{1}{6}(12)^3$$

$$= \mathbf{144 \text{ m}}$$

(iii) Acceleration $a = \frac{dv}{dt} = 6 - t$

When $t = 8$, $a = 6 - 8$

$$= \mathbf{-2 \text{ ms}^{-2}}$$

7. **Topic: Applications of Differentiation (Gradients, Tangents & Normals)**

Given $y = \frac{\sin x}{2 - \cos x}$, $0 < x < \frac{\pi}{2}$

$$\frac{dy}{dx} = \frac{(2 - \cos x)\cos x - \sin x \cdot \sin x}{(2 - \cos x)^2}$$

$$= \frac{2\cos x - \cos^2 x - \sin^2 x}{(2 - \cos x)^2}$$

$$= \frac{2\cos x - 1}{(2 - \cos x)^2}$$

Tangent to curve is parallel to the x-axis

$$\Rightarrow \frac{dy}{dx} = 0$$

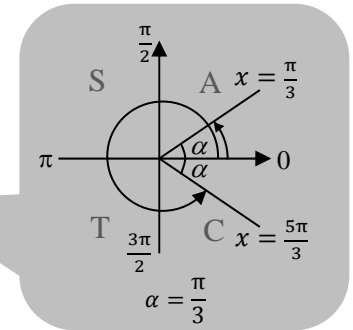
$$\frac{2\cos x - 1}{(2 - \cos x)^2} = 0$$

$$2\cos x - 1 = 0$$

$$\cos x = \frac{1}{2} \Rightarrow \text{Basic angle } \alpha = \frac{\pi}{3}$$

$$x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ (reject } \because 0 < x < \frac{\pi}{2})$$

$$\therefore x = \frac{\pi}{3}$$



8. **Topic: Further Trigonometric Identities**

(i) For $\sin 3x + \sin x = 4 \sin x \cos^2 x$

Factor Formula:

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

L.H.S.: $\sin 3x + \sin x$

$$= 2 \sin \frac{3x+x}{2} \cos \frac{3x-x}{2}$$

$$= 2 \sin 2x \cos x$$

Double Angle Formula:

$$\sin 2A = 2 \sin A \cos A$$

$$= 2(2 \sin x \cos x) \cos x$$

$$= 4 \sin x \cos^2 x = \text{R. H. S. (Shown)}$$

(ii) $\sin 3x + \sin x = 2\cos^2 x$

From (i): $\sin 3x + \sin x = 4\sin x \cos^2 x$

$$4\sin x \cos^2 x = 2\cos^2 x$$

$$4\sin x \cos^2 x - 2\cos^2 x = 0$$

$$2\cos^2 x(2\sin x - 1) = 0$$

$$\cos x = 0$$

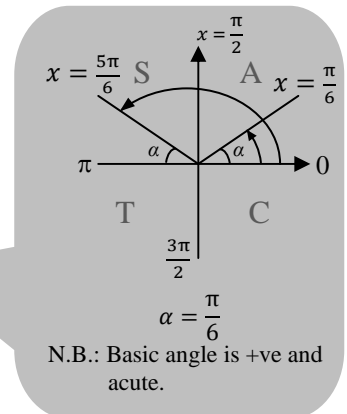
$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{2}$$

Basic angle $\alpha = \frac{\pi}{6}$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$



9. Topic: Simultaneous Equations

Let Ann's age be x and Betty's age be y .

$$x^2 - 2y^2 = 6(x - y) \dots\dots\dots (1)$$

$$x + y = 5(x - y)$$

$$x + y = 5x - 5y$$

$$6y = 4x$$

$$y = \frac{2}{3}x \dots\dots\dots (2)$$

$x^2 - 2y^2$: "... twice the square of Betty's age subtracted from the square of Ann's age"
 $= 6(x - y)$: "... equal to 6 times the difference of their ages."

Sub (2) into (1): $x^2 - 2\left(\frac{4}{9}x^2\right) = 6\left(x - \frac{2}{3}x\right)$

$$\frac{1}{9}x^2 = 2x$$

$$x^2 - 18x = 0$$

$$x(x - 18) = 0$$

$x + y = 5(x - y)$:
 "... sum of their ages is equal to 5 times the difference of their ages"

$x = 0$ (reject as Ann is older than Betty) or $x = 18$

When $x = 18$,

$$y = \frac{2}{3}(18) = 12$$

∴ Ann is 18 years old and Betty is 12 years old.

10. Topic: Quadratic Equations & Inequalities

(a) $ax^2 + 5x + 2 > 0$

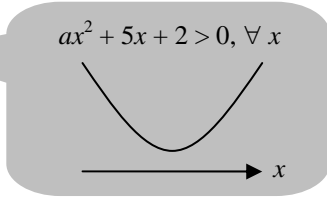
$$\Rightarrow b^2 - 4ac < 0 \text{ (no real roots)}$$

$$25 - 4a(2) < 0$$

$$25 < 8a$$

$$a > \frac{25}{8}$$

$$> 3\frac{1}{8}$$



∴ smallest integer $a = 4$

(b) $-5x^2 + bx - 2 < 0$

$$b^2 - 4ac < 0 \text{ (no real roots)}$$

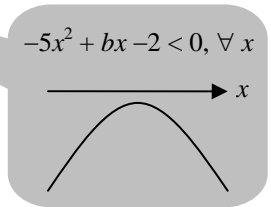
$$b^2 - 4(-5)(-2) < 0$$

$$b^2 - 40 < 0$$

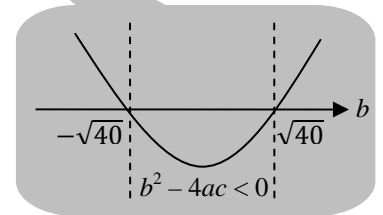
$$(b - \sqrt{40})(b + \sqrt{40}) < 0$$

$$-\sqrt{40} < b < \sqrt{40}$$

$$\Rightarrow b > -\sqrt{40} \approx -6.32$$



∴ smallest integer $b = -6$



11. Topic: Binomial Expansions

$$(i) (r+1)^{th} \text{ term of } \left(x + \frac{k}{x}\right)^7 = \binom{7}{r} x^{7-r} (kx^{-1})^r$$

$$= \binom{7}{r} k^r x^{7-2r}$$

In expanding $(a+b)^n$,
 $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$\text{Coefficient of } x^3 = \binom{7}{2} k^2$$

$$= 21k^2$$

$x^3 = x^{7-2r} \Rightarrow r=2 \rightarrow T_2$

$$\text{Coefficient of } x = \binom{7}{3} k^3$$

$$= 35k^3$$

$x^1 = x^{7-2r} \Rightarrow r=3 \rightarrow T_3$

Equating coefficients of x^3 and x ,

$$21k^2 = 35k^3$$

$$3k^2 = 5k^3$$

$$5k^3 - 3k^2 = 0$$

$$k^2(5k - 3) = 0$$

$$k = 0 \text{ (reject) or } k = \frac{3}{5}$$

$$\therefore k = \frac{3}{5}$$

$$(ii) (1 - 5x^2)\left(x + \frac{k}{x}\right)^7 = (1 - 5x^2) \left[\binom{7}{0} x^7 + \binom{7}{1} kx^5 + \dots \right]$$

$$= x^7 - 35kx^7 + \dots$$

$$\therefore \text{coefficient of } x^7 = 1 - 35k$$

$$= 1 - 35\left(\frac{3}{5}\right)$$

$$= 1 - 7(3)$$

$$= -20$$

Terms beyond the 1st 2 terms of $\left(x + \frac{k}{x}\right)^7$ are ignored as they do not form x^7 terms when multiplied by $(1 - 5x^2)$.

12. Topic: Linear Law

$$(i) \text{ Given } y = kb^x,$$

$$\lg y = \lg(kb^x)$$

$$\lg y = \lg k + x \lg b$$

$$\lg y = (\lg b)x + \lg k$$

Letting Y be $\lg y$ and X be x , the graph of $\lg y$ against $\lg x$ is a straight line

$$Y = (\lg b)X + \lg k$$

$$= mX + c$$

From the graph,

Gradient m :

$$\Rightarrow \lg b = \frac{1.3 - 0.8}{0 - 11}$$

$$= -\frac{1}{22}$$

$$b = 10^{-\frac{1}{22}}$$

$$\approx \mathbf{0.90 \text{ (2 sig. fig.)}}$$

When $X = 0$, Y -intercept = 1.3

$$\Rightarrow \lg k = 1.3$$

$$k = 10^{1.3}$$

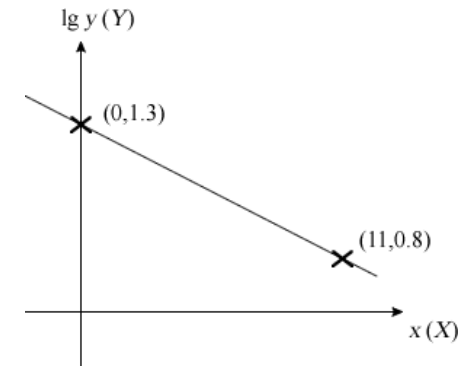
$$\approx \mathbf{20 \text{ (2 sig. fig.)}}$$

$$(ii) \text{ From (i), } y = 10^{1.3} \left(10^{-\frac{1}{22}}\right)^x$$

When $x = 8$,

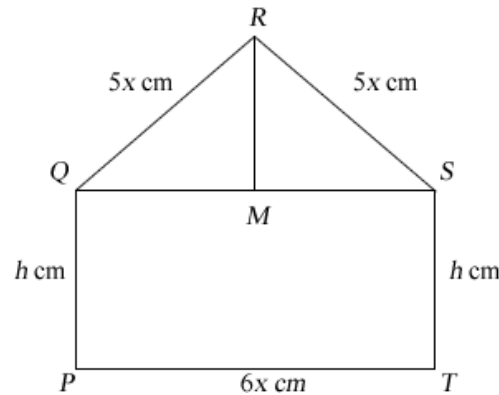
$$y = 10^{1.3} \left(10^{-\frac{1}{22}}\right)^8$$

$$\approx \mathbf{8.64}$$



13. Topic: Applications of Differentiation (Maxima & Minima)

(i)



Perimeter of the window = 360cm (given)

$$\begin{aligned} 5x \times 2 + h \times 2 + 6x &= 360 \\ 5x + h + 3x &= 180 \\ h &= 180 - 8x \end{aligned}$$

Let RM be the height of $\triangle QRS$.

$\Rightarrow M$ is the mid-point of QS . ($\triangle QRS$ is isosceles)

Using Pythagoras' theorem,

$$\begin{aligned} RM &= \sqrt{(5x)^2 - (3x)^2} \\ &= 4x \text{ cm} \end{aligned}$$

$$\begin{aligned} \therefore A &= \frac{1}{2}(6x)(4x) + (h)(6x) \\ &= 12x^2 + (180 - 8x)6x \\ &= \mathbf{1080x - 36x^2} \text{ (Shown)} \end{aligned}$$

Area A = Area of $\triangle QRS$ +
Area of rectangle $PQST$

To show that $A = 1080x - 36x^2$,
express all unknowns i.e. h and RM
(height of $\triangle QRS$) in terms of x .

(ii) $\frac{dA}{dx} = 1080 - 72x$

When $\frac{dA}{dx} = 0$,

$$\begin{aligned} 72x &= 1080 \\ x &= 15 \text{ cm} \end{aligned}$$

When $x = 15$ cm,

$$\begin{aligned} A &= 1080(15) - 36(15)^2 \\ &= \mathbf{8100\text{cm}^2} \end{aligned}$$

(iii) $\frac{d^2A}{dx^2} = -72$

$$\Rightarrow \frac{d^2y}{dx^2} < 0$$

\Rightarrow The stationary value of A is a maximum.